THE

MATHEMATICAL

GAZETTE

CONTENTS.

THE ORIGIN OF MATHEMATICS IN GREEK CULTURE. *

By Prof. W. M. Edwards.

This paper, like some characters in the Bible, has already been twice baptised; it began by being called "The Atmosphere in which Greek Mathematics arose"; that, however, is, as Lewis Carroll would say, only what it was called; its name turns out to be "The Origin of Mathematics in Greek Culture". I may say that neither of these designations originated in my own brain; the former I gave my consent to; the latter has been sprung upon me. However, it does not signify; because what the paper really is (if I may quote once more from that great metaphysician), will probably turn out to be something else. Let me state at once that, though I am interested in mathematics in a crude and general way, as I had reason to be in my profession, I have never been accustomed to go to the so-called Greeks for information about it (or should one say "them"?); though it seems quite clear that it was the Greeks that started the whole business. And as for the Greeks themselves, it is not their culture or their atmosphere which interests me so much as the fact that they spoke, and still speak, a very interesting variety of the same Indo-European tongue that we use ourselves; one which lends itself to scientific treatment. However, that is not what you have asked me to speak about; so I must try to generate a pressure of as many atmospheres as I can. I see first of all that it is an affair of background—if I may adopt a happy term often employed in relation to matters of culture by the Professor of Mathematics. The mathematician fortunately needs, for his own specific productions, nothing but the background of his own consciousness. It is this beautiful and significant fact which, to anticipate a little, seized on the imagination of such a man as Plato, and caused him to accord almost divine honours to your science. The mathematician himself may, however, stand in much need of a background when he is contemplated by ordinary people; without it he would be altogether too stark an object. In the present case we evidently need two backgrounds, one of space and one of time—or rather a space-time complex; but as I have not learnt to draw that, I have contented myself with producing a map of sorts; I hope you will excuse its shortcomings. I mean it to remind you of the great space covered by the Hellenic race, or by its sphere of influence, at the time of its maximum expansion; this ranged, in fact, from the Straits of Gibraltar to the Indus; in some regions a mere coastal fringe; in others going deep into the continent. In time, if we limit this to the period popularly known as that of the "Old Greeks", this civilisa-

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tion covers a thousand years or so ; for some purposes, such as that of language, it covers three times that. The race itself, at least in its early stages, is only entitled to the name in virtue of a unity imposed on what was originally a variety of races ; and even so, we find a bewildering multitude of tribes, mostly hostile to each other, speaking dialects which are closer akin, certainly, than English and Welsh, but are often wider apart than English and Aberdonian. When we add the possibilities of considerable racial mixture with non-Hellenic peoples on the fringes of the domain—a fact which is attested by Greek writers—then it makes it very difficult to say with certainty in any given case that we have bottled a genuine specimen of Greek atmosphere. The centre of dispersion of this interesting people really was Greece, though they did not call it that ; wave after wave of immigrants and conquerors kept pouring down into the peninsula from the north at various times during the second millennium before Christ ; and at the same time the populations within it began to send emigrants eastward across the Aegean Sea to the coast of Asia Minor and as far as Cyprus, regions where their language has been spoken ever since. I would call your attention especially to the people who proceeded to occupy the western fringe of Asia Minor. They were not all of one blood ; those who settled in the extreme north and south of that fringe remained mathematically undistinguished ; but the central strip, from Smyrna to Miletus, together with the islands which lie close to that coast, namely Ionia, was occupied by a people who are very important for our present purpose. They called themselves Yawones, which is one of the great names by which the Hellenic race has been known by outsiders ever since. The Hebrew writers of the Old Testament called them Javan ; the Persians, their enemies and oppressors, called them Yawnau ; Oriental peoples to-day call the Greeks Yunani, which is a slightly different form of the same word. It is a legacy of three thousand years ; a memory of the first contact with the East of the most enterprising and vigorous section of the Greek race. They call themselves Hellenes to-day, but it took them some time to do it ; we might compare our own shyness in the matter of our national name. We only call ourselves " British " in the visitors' books of hotels (except those of us who proudly write " Scottish ") and " Britons " only in moments of lyrical exaltation. There was something of the same religious feeling about the name of Hellas.

To return to the Ionians, however. These vigorous early emigrants from the mainland of Greece are shown, by their dialect and traditions, to be first cousins of the Athenians who remained in their original position in Attica, roughly due west of Ionia ; the colonisation must have proceeded from that region. We are right in the importance which we attach to the Athenian civilisation ; but it is as well to remember that the Ionian preceded it in all branches—art, literature and science ; and to some extent inspired and instructed it. The geographical position of the Ionian fringe is a very precarious one ; its shape and topographical nature has always tended to prevent political unity. Its history has been chequered beyond most ; when the West has been strong, the menace of the Eastern " barbarian " has been pushed back ; when the West has been weak, or indifferent, the Oriental has either reduced it to vassalage or else has attempted to destroy it in blood and fire, as in the miserable events of nine years ago. But at the period I am speaking of, that is to say, in the first phenomenal flowering of its civilisation, it enjoyed certain advantages in this respect. There was, for the moment, no great foreign power in effective control of Asia Minor with Westward ambitions, and Ionia was accordingly able to develop. This is the period of the poet Homer—by whom you will allow me to mean the final arranger of a great mass of early poetic material. Please do not think of Homer as an old blind man in a nightgown ; he appears to me rather as a young man ; dressed, I have no doubt, in the height of contemporary fashion, and with eyes of remarkable

keenness, especially for Nature and all her phenomena, which is as much as to say that he was an Ionian. At this period, also, occurs an event which is one of the greatest in the history of Western civilisation ; truly a *magnus partus temporis*. The clumsy Phoenician syllabic signs, or some set closely akin to them, are taken over and transformed, by a series of simple but highly ingenious touches, into the business-like set of phonetic symbols which are now used all over the civilised world (including Russia)—the Ionic letters. Literature is let loose. It would take me too far from my subject to describe how the Latin and all other white men's alphabets have been developed out of this ; but it may be called a considerable set-off for what must appear, in the present connection, as a grave defect in the Hellenes, namely, that they never evolved a workable set of numeral signs.

We know comparatively little about the early social and domestic life of the Ionian people ; not so much as, for instance, about the Athenians, or even the Spartans. It is probable that they were mixed to some extent with non-Greek blood by the time they come into the fuller light of history ; " cities of mingled Hellene and barbarian strain," the poet Euripides calls them—meaning in that context nothing derogatory by the term " barbarian ", but simply " of foreign speech ". We know them chiefly by their achievemen's in commerce and colonisation, and by what remains of their poets and prose-writers ; most important among these, for our present purpose, being the series of great men who sowed the first seeds of so many sciences. The one phenomenon was to some extent the cause of the other ; there was the need for keeping in touch with foreign customers, and the natural interest in geography which was connected with the need for undertaking long over-sea voyages for the foundation of colonies. Thus they were encouraged in the study of the earth's surface, no less than in that of the heavenly bodies. The historian Herodotus mentions an Ionian map of the Eastern world, engraved on bronze, which was brought to Old Greece by a messenger in the sixth century ; evidently an unheard-of wonder for the king of Sparta. That was no doubt a crude affair, based on the dead-reckoning of ships' captains and the days' marches of travellers by land. It was a ridiculously short time, when we consider the period of the world's history with which we are dealing, before men of the same race were measuring the degrees of a spherical earth by astronomical observation. A centre of this interest was the great trading city of Miletus, at the southern end of the Ionian strip ; by the middle of the sixth century she had studded the coasts of the Black Sea with her colonies or trading-posts ; the names of several of them still endure. She had her factories in Egypt as well. Flax and metal were imported from these places in return for the sheep's wool grown on the Ionian pastures. The Egyptian contact is probably important for the beginnings made upon geometry. It was evidently no more than a bare hint that was received from this quarter ; as a later chronicler states, " It is said to have been invented in Egypt for the mensuration of land ; as the periodical floods used to obliterate the boundaries between holdings, and these had to be rectified ". There is grave reason to suppose, I believe, that the Egyptians were in the habit of treating inconvenient obtuse angles as angles of 90°. In any case, we probably have to deal here with as radical and decisive an improvement, on the part of Egypt's Ionian pupils, as we have seen before in the matter of the alphabetical signs. More than one of the Ionian " philosophers ", people, that is to say, interested in the observation and explanation of Nature, are said to have visited Egypt. Earliest in time is the great Thales of Miletus, who, by the way, was probably half a foreigner by birth ; his father's name is not Greek. He is remembered for having been in a position to predict an eclipse of the sun in 585 B.C., on account of which he is generally supposed to have drawn upon Babylonian records. There is no independent means of verifying this ; isolated Greeks from the coast occasionally found their way inland as far as the great Meso-

potamian cities ; but such contacts, if they occurred, probably took place by way of the Syrian coast ; such information may even have reached Egypt. Thales, like others of his kind, interested himself not only in astronomy and geometry, but in all aspects of physics, as we may call it generally ; he was the father of one of the premature speculative theories about the constitution of matter, now for the time being consigned to the lumber-room. These theories are pathetic enough in their way, even paradoxical ; the Greek mind, refusing to admit the limitations which circumstances forced upon it, seems to desert for the time being the solid ground of reason and observation which is its true field of action, and to plunge desperately into the unmeasureable and incalculable, which is so detestable to its instincts. One good thing, at any rate, can be said about these speculations ; they were conducted in an atmosphere that was absolutely free. They were completely unhampered by any sort of tradition. There were no ancient seats of learning with fossilised curricula or ancestral prejudices ; and as for religion, the man in the street, or on the farm, might amuse himself with propitiation of gods or godlings, or might pour blood into trenches for ghosts to drink ; the brotherhood of learning was monotheistic or agnostic—whichever one prefers to call it. The influence of Ionia was very soon felt wherever the Greek language was spoken ; and there are few of these great men who did not, at one time or another, travel to some part of the Greek Mediterranean world, often causing new centres of study to spring up in places where they might never have done so without the Ionian inspiration.

A notable example of this missionary work, as we may call it, is Pythagoras. Samos was his home ; the island which lies off the coast close to the roadstead of Miletus. It is more than usually difficult, in the case of this man, to decide precisely what contribution he made to science ; we sense a great and commanding personality ; but legend has gathered as thickly round it as it has round that of Merlin or of Moses. He is said to have travelled in Egypt, and to have assimilated all the wisdom of Chaldaea. We have a glimpse of him visiting the great athletic festival of Olympia, when his reputation had already gone abroad ; the crowd salute him with the title of Sophistes or " Professor " ; literally, " one who undertakes to make people wise " ; he replies, " Call me rather Philosophos, the lover of wisdom ". Getting into trouble in his native island, which is by this time under a tyrant, he migrates to Southern Italy, where he settles down and proceeds to gather pupils and adherents round him ; there is very soon a flourishing school among the South Italian Greeks, the influence of which spreads to Sicily and later makes itself felt as far as the old Greek mainland. He adopts an austere and retiring way of life, in which we may suppose there is not a little design, not to say affectation ; there is a probationary silence-rule of five years imposed on persons who attend for instruction ; this may be reduced to two in cases of natural taciturnity, which would probably be rare amongst this people. He is respectfully and affectionately known by his pupils as ", Himself " ; " Himself said it " is accepted for examination purposes as equivalent to a text-book reference or a rigorous demonstration.

The paternity of many of the discoveries in mathematics, especially in geometry, is made doubtful owing to the habit of collecting, revising and arranging the work of predecessors, without any acknowledgment being thought necessary ; as indeed is quite natural, unless a man is fortunate enough to get his name attached to some method or theorem in such a way that one can hardly avoid printing it at the head of the paragraph in leaded type. In the case of Pythagoras, there seems to be no harm in allowing him the 47th Proposition, which tradition has always associated with his name ; and with having encouraged the investigation, in a geometric sense, of equations in the second degree, bound as he was in the fetters of whole numbers. He may also be credited with the discovery of the mathematical ratios govern-

ing harmony ; these were based, as you know, on the length of the strings which gave out the musical notes. This was calculated to make an immediate appeal to a man of his race—even to launch him on a path of dangerous mysticism. There are few languages that are so saturated as the Greek with metaphors drawn from musical harmony ; we have inherited many of them, either by way of Latin or by direct contact with Greek literature. They express something which is a fundamental instinct in the Greek mind : that is to say, there is a proper and natural Rhythmos, or balance of form, in every object of perception, whether it be an object of sense, in which case the rhythm will be perceived by the eye or ear, or an object of thought, when it will be perceived by the mind alone. If an object has this rhythm, it is right ; if it has not, it is wrong. If it is wrong, it may either be capable of being " rhythmised " or tuned ; or it may not ; in which case it is a delusion and a lie—a part of Chaos. The possession of rhythm also entitles an object to the name of Beautiful ; here we seem to arrive at the dangerous border-line between Intellectual and Aesthetic appreciation. I am not quite certain what the latter term means ; but I feel fairly sure that in the case of the Greeks it is difficult to put our finger on any emotional phenomenon and to say " Intellect had no part in this ". As to the Beauty of the object of thought, every mathematician will know at once what I refer to. He who has never felt a mystical thrill when contemplating some exquisitely ingenious method or theorem is no mathematician at all, but only a hireling. Or to chess-enthusiasts, of whom I hope there are many present, I would suggest the example of a brilliant and economical " problem ".

Anyhow, Pythagoras would seem to have wandered far on this path. Other quaint things are recorded of him : as, that he indulged in theories of metempsychosis and other ghostly matters ; that he constructed a mirror, which, after tracing upon it certain characters in human blood, he directed towards the orb of the full moon, upon the surface of which the same symbols immediately became visible. He had an underground cave, into which he used to retire for long periods of work and meditation, refusing all communication with the outside world. It is here that is probably to be located the " Dream of Pythagoras ", described in the so-called Golden Verses of his disciple Polyheuron of Phaeacia. Pythagoras, he says, while toying with the chords of a lute, became drowsy, and eventually fell asleep to the pleasant accompaniment of the Major Third. On beginning to dream properly, he seemed to find himself surrounded by all the objects of sense that he knew in waking, including the sun, moon and planets in their courses ; in which it had always been his ambition to discover the application of mathematical—or, as he put it, geometrical—relations, similar to those which he had found to obtain between the strings of the lute. This had been impossible for him to verify in real life, because he had not the means of conducting suitable experiments with them, or did not yet see how to set about the business ; in fact, " he was in the stern grip of necessity ", as the author remarks. In his dream, however, it appeared to him that all his difficulties were smoothed away by a power greater than his own ; he was able to investigate and reduce to law the harmonies of the whole physical universe, beginning with the majestic movements of the heavenly bodies and ending with the smallest particles of matter, which, with his brother physicists, he conceived to be ultimate. These laws turned out to be few in number and surprisingly simple ; he almost regretted their simplicity. He perceived that they were, and could in their nature be, nothing but the work of mind ; he therefore acknowledged that God was the Great Geometrician of the Universe. This universe seemed to him, owing to his preconceptions, to be finite and externally bounded by Chaos, or the Outer Infinitude, which presented itself to him as a surrounding wall, constructed of adamant and impenetrable to the touch. At this point he ceased for a while to dream coherently, but continued to enjoy a beatific coma in which

his only conscious impression was that he was identified for the time being with the musician Orpheus in a previous incarnation—this was a point in his theory of metempsychosis—and that the whole of creation was moving in time to his music. He was, in fact, though unconsciously, beginning to compete with the Geometrician.

In due course he rose to an intermediate stratum of consciousness and commenced the second part of his dream, which was ushered in to the sound of the Dominant Seventh. He seemed to emerge again into the world of sense with its orderly collection of objects ; these he began to handle once more, full of sensuous delight in the symmetry of their forms, which was now perfectly intelligible to him, and in the comfortable solidity of their touch. But at this point an unquiet spirit arose in him ; he felt that he was possessed of divine powers of analysis which had hardly been tested so far ; what if the ultimate should not prove to be ultimate, and the boundary wall turn out to be non-existent ? So it was ; as he plunged into the work, the solid, liquid and gaseous matter, which he had thought to be the end of things, kept resolving itself into smaller and smaller particles ; Anaxagoras's Atoms proved to be no barrier ; he passed through and beyond them almost without perceiving that he had done so ; and the new particles were still held together by mathematical relations. At last he found that he had rubbed away all matter whatsoever to the vanishing point ; it was in fact non-existent ; for he was by this time endowed with such powers of vision that, if it had still had any appreciable existence, he would have certainly been able to detect it. It was gone ; and in its place there trickled through his fingers an infinitely fine and living gauze, the pure tissue of thought. Filled with a great fear, and an insane desire to touch something solid, he approached the boundary wall of Chaos, stretching out his hand. As he got near to it, it now seemed to him to have a transparent and reflecting quality ; it was, in fact, a great mirror, similar to that on which he had traced the characters of blood. In it he was not surprised to see his own figure approaching him ; what seemed more unusual, however, was that he appeared to meet and coalesce with it, and, furthermore, to proceed on his way without encountering the slightest opposition ; most inexplicable of all, though he seemed to continue moving in the same direction, he soon became convinced that he was returning to the point he had originally started from. With a terrible shock he realised that he was everywhere and nowhere at once, in a universe of thought created entirely by himself. " ' Himself said it ' ", he quoted, remembering his pupils, whose very existence had now become questionable to him ; " say rather, ' Himself made it ' ". It was doubtless after this dream that, as Iamblichus relates, the philosopher emerged from his subterraneous cavern " with a grim and ghastly countenance, and declared in the assembly of the people that he had been in hell ".

Pythagoras, or as much of him as is not legend, lived about six hundred years before Christ ; and Ionia was not to enjoy more than another century of freedom from oppression or conquest ; but he was by no means the last scientific missionary to bring light from Asia. Consider the small city of Cnidos, more than two centuries later. It stands on a promontory running out into the Ægean Sea, south of Miletus, in what had previously been a Dorian or non-Ionian region, and perhaps was so still to some extent, though Ionian influence had penetrated far down the coast. The citizens of Cnidos are met for a religious festival ; no less an event than the dedication of a new statue of the Goddess of Love, expressly executed for their famous shrine of her by the greatest of contemporary Athenian sculptors, who took for his model an equally distinguished courtesan. Two young men of Cnidos are in the congregation ; one of them nameless, but intensely interested in the proceedings ; the other preoccupied—Eudoxos the name of him. The ceremonies are ended, and night falls ; Aphrodite of Cnidos remains alone on

her pedestal. At midnight the young man without a name comes, feeling his way fearfully through the darkness of the shrine, to embrace the divine marble ; the Cnidians say that to this day it bears the marks of the passionate sacrilege. The other, Eudoxos, is lying awake in his garret considering the matter of incommensurables and the generalised theory of proportion. If this were a real paper on Greek mathematics, it would be necessary for me to speak at length about Eudoxos ; his methods of exhaustion (which I must not imitate) ; his theoretical solution of the planetary orbits ; his contributions to fundamental method in the establishment of the rigour of proof. Fortunately for me, however, it is not so ; and fortunately for you there is at least one real mathematician in the University who is deeply versed in these historical questions, and who will be delighted, I am sure, to satisfy the cravings of any genuine enquirer. Eudoxos is one of the big milestones ; I will leave him at that.

The mention of him, however, reminds me that there are other centres of polite learning besides Ionia ; Eudoxos migrated, in fact, to Athens, where he was drawn into Plato's circle. As I have hinted before, at the time of Ionia's first amazing burst of vitality Attica was still in a state of comparative backwardness. The people were agriculturists in a small way, or the majority of them ; they raised some corn in their rather limited plains, to which was added the cultivation of the olive tree ; the uplands afforded pasturage to goats. The people were intensely conservative, like most agriculturists ; they endured until the middle of the sixth century vicious economic conditions which brought the country to the edge of ruin, from which it was rescued by the help of a temporary Mussolini. He, unlike his Italian counterpart, retired gracefully from the scene after initiating various political and economic reforms, including regulation of the currency.

Art there certainly was among this people, if of a primitive kind ; and the reformer himself composed some honest but uninspired verses, the first literary monuments of his country. The dynasty of tyrants who followed him made at least a start upon systematic education in the State, and encouraged the study of Homer. Ionian influence, in fact, was strongly at work in both these departments ; though of Ionian science there was hardly a trace. It seems incredible that this was the state of things hardly more than a century before the age of Pheidias and Euripides. It is probable that no similar period could be found in human history showing anything like the same upward curve ; the spark which ignited it (if I may juggle with metaphors) was probably that blessing in disguise, the Persian invasion.

The question immediately before us, however, is whether the Athenians were real honest mathematicians. I may say at once that I have the very gravest doubts about this. They were essentially humanists ; that is to say, they found their chief interest in social intercourse, politics (national and parish-pump), business and litigation ; everywhere and at all times dissecting the characters of their neighbours ; even occasionally dissecting their own. Much of the physical nature which the sacrilegious Ionian thought it his business to probe into was to them divine—or at any rate, that was their traditional convention. It appears to be a historical fact that a public jury went through the form of condemning to death one of their most enlightened humanists, amongst other charges on that of importing and encouraging such speculations ; it goes without saying that he actually abominated them, and was at pains to convince the jury of this, but apparently without success. Being a humanist, and not a little of a humorist too, he also put them to the trouble of carrying out their sentence, although every door of his prison was studiously left open.

But enlightenment eventually won the day, even at Athens. By the end of the fifth century it was at any rate possible for distinguished outsiders to visit the city in comfort ; in fact they did so in considerable numbers ; the

Athenian love of conversation threw down all barriers. Their practical sense also led them to take advice in some matters ; as when Meton, no doubt using information collected further east, set about a rather half-hearted correction of one of the most appalling calendars which a civilised people has ever suffered under. It is probable, too, that they distinguished somewhat sharply between mathematics and physics, to the disadvantage of the latter. Pure mathematics—or the purer the better ; astronomy would certainly not be excluded, but chiefly because it was concerned with noble and divine objects, not with cosmic mud. Technically their mathematics did not yet go far ; Plato speaks with rapture of the regular solids, and looks forward to the time when research in geometry will be endowed by the State ; he is at pains to demonstrate his acquaintance with the harmonic series, and so forth. It needed the arrival of a Eudoxos to push the matter further. This, however, is not to deny the enormous impulse which was given by Plato to all branches of intellectual progress, including that in mathematics. This impulse, in my humble opinion, far transcends in significance any technical advance that was subsequently made. And there is no doubt about Plato's attitude towards mathematics. In his arrangements for education in his ideal State, as you are aware, he gives to this science a position practically corresponding to what we should regard as the whole of a university course, having regard, that is, to the age at which it is to be undertaken. For him it is the great propaedeutic—the study which is to form and strengthen the mind for dealing with metaphysics and moral philosophy, which are to be the work of the grown man—or such men as are privileged to complete their education. It is clear at once that this is not in virtue of any " application " of the science, in the popular sense ; it is chiefly because mathematics means exercise in abstract thought. On the Platonic view every individual is potentially a mathematician ; and not only that, but he is potentially a Newton, and that, too, without any adventitious aid ; for there is no step in mathematics which is not the simple logical outcome of previous steps. You will recall, no doubt, the experiment of Socrates on the slave-boy, whom he causes to demonstrate his acquaintance with a simple proposition in plane geometry which he has never seen before. A similar story is told of the infant Pascal, who, I understand, proceeded to evolve the whole of the first book of Euclid after being shown the Pons Asinorum. It is a beautiful thought for me that if Professor Milne and I had been planted at birth on two adjacent desert islands, and fed on goats' milk by deaf-mutes, he and I might now have been, mathematically speaking, equal and opposite, instead of being separated by the enormous gulf which actually yawns between us. It is equally comforting to reflect that if the greatest contemporary physicist had been similarly planted on a third island, he would at this moment be in much the same position, physically speaking, as the deaf-mute, or even the goat.

However that may be, it is evidently this aspect of mathematics which impressed Plato and those who thought like him. If asked why he did not require education to consist entirely of mathematics, he would probably reply that the world of logical thought, which was for him the real world, includes a great deal that cannot be brought within the compass of that science. For logical inference is not concerned exclusively with objects which can be weighed, counted or measured. It is true of many sciences—history, for instance, or linguistic, or biology—that they are concerned with a multitude of logical constructions into which the mathematical aspect need not enter at all ; when it does so, it is usually statistical. Finally, metaphysics, which for Plato was the crown of all education, consists entirely of logical inference of the non-mathematical type. For all that, Plato was not a little affected by the Pythagorean mysticism with regard to the nature and possibilities of number ; and it is likely that the bounds which he would set to the sphere of the mathematical concept are not quite the same as they appear to us at

the moment. As for its almost complete victory in the physical sphere, he would regard that as the most natural and proper thing in the world.

The Platonic school undoubtedly gave a great impulse to the historical development of mathematics. This influence was both direct and indirect. Many of the Platonic dialogues contain exercises in the " analytical method " ; an initial assumption (if I am stating the process correctly) of the proposition to be proved, followed by an investigation backwards, step by step, of the suppositions involved in this, until a known premise is reached whose truth or falsity can be tested. It sounds simple now ; but it required to be thought of ; and it is probably on Plato's suggestion that the method was developed by more distinguished mathematicians. The science was, in fact, considerably advanced in the academic circle by such men as Eudoxos, Leon, and Menaichmos, the founder of the geometry of conic sections.

By the time of Plato's death the sands were already running out for the political prestige of Athens, and indeed of Old Greece in general. It is true that Athens continued, partly by courtesy, to retain her traditional position in the intellectual world—which, it is as well to remind ourselves from time to time, contains a few non-mathematical elements ; she did so right down through the period of Roman domination. The shadow of that glory, like that of the oracle of Delphi, continued to haunt the banks of the Ilissos long after all reality had departed. Honest bourgeois, like Cicero's father, prided themselves on enabling their youthful progeny to attend its talking-shops ; these young men came away with the *cachet*. Its professors were welcomed in remote barbarian cities, and were given free platforms before enthusiastic but uncomprehending audiences. It is related that the elder Cato was shocked to the depths of his Roman soul by an Athenian performer who devoted one lecture to the establishing of a proposition and the next to its destruction ; evidently not a proposition in mathematics. He brought before the Senate a bill for the exclusion of such people as undesirable aliens.

This, however, was long after the period I am speaking of—the fourth century before Christ. We are now faced by one of the great revolutions of history ; the end of one period, which, in spite of later developments, I cannot help feeling is the really great and creative period for your science ; and the beginning of one that was very unlike it. The Hellenic world, over a small part of which we have been roaming—for there were a host of other mute inglorious Greeks who thought themselves quite important—was up till now divided into a multitude of small republics, having their own dialects and their own traditions, which were neither Athenian nor Ionian. The sentiment of a common Hellenic bond was planted firmly enough at the back of their minds ; but it had required an athletic meeting to make it articulate, and a world-war to call it into effective existence. These little cities, with their oligarchic or democratic systems of government, were intense centres of social and political life. The very smallness of them enabled their popular assemblies to be primary, and not representative ; the whole body of enfranchised citizens could, and often did, attend in a place where they were supposed to be able to hear the voice of a single orator. Except in the more technical business of the law-courts, as practised for example at Athens, these assemblies were accustomed to vote by acclamation, often developing into execration. It was clearly not always an easy matter to assess the division of opinion. The practical, but otherwise non-mathematical Lacedaemonians, not being acquainted with the microphone, devised the plan of placing a select committee of acoustai, or listeners, in a dark room adjoining the popular meeting-place ; the average result of their individual impressions was taken as decisive.

It was, on the whole, a healthy atmosphere ; one which tends to encourage strong and original intellectual work, provided that the people is naturally gifted in that way. But political freedom of this type was not to last. The

city-states had had their innings. Their vitality was ebbing; and when the next great external menace made its appearance, they were no longer in a condition to sink their differences and unite against it, as they had done a century and a half before. This time it came from the non-Greek country of Macedonia, whose rulers had for some time cast greedy eyes on the peninsula, and had prepared the way by political intrigue, always an easy tool to use against the Greek cities. The Macedonians, though foreigners, had been steeped for some time in Greek culture and atmosphere; their rulers had seen to it. Even their personal names, those at least of historical characters, had largely become Greek in form; their rulers were called Philippos," lover of horses ", and Alexandros, " champion of heroes ". Greek political freedom vanished, almost unnoticed, in the smoke of incense offered to the young conqueror, the " new Heracles ".

Alexander allowed Greece no time for second thoughts. Without resting a moment he betook himself to the march of conquest into the far interior of Asia which was only to end with his death. The multitude of mercenaries and adventurers of all kinds who now began to leave Greece for the new territories hastened the depopulation of the old country. The new empire extended to Egypt in the south, and eastwards as far as Bactria and Afghanistan; there was even a campaign into north-west India. Townships and military stations were established at intervals over this vast region—nominally Macedonian, but actually filled to a great extent with Greek settlers. The Greek language was in official use everywhere, and formed the natural medium of communication among all classes. It was a new and universal variety of the speech, supplanting the old local dialects; based chiefly on Athenian, with an admixture of Ionic, these being the dialects which were in the strongest position for a number of reasons. Others went to the wall; and it is from this *lingua franca* that all the varieties of modern Greek have arisen; the Old Testament was translated into it, and the New composed in it. It became the tongue of the new " Greece of the dispersion ", and spread itself over Alexander's empire in response to a universal need; the new settlers soon lost the old *patois* which they had brought with them from their native cities.

At the same time, they rapidly lost any sentiment of attachment to a mother-city in the home country; a feeling which had, on the contrary, remained very strong during the earlier period of Mediterranean colonisation. If the new Greek was a citizen of anything, he was a citizen of one of the new cosmopolitan cities, unexampled in the history of the West, such as Alexander's name-foundation at the mouth of the Nile; cities greater in extent than any that the Greeks had lived in before; counting their hundreds of thousands of population where the old Greek towns had hardly counted tens. The empire broke up at the founder's death into large areas, each of them ruled over by a royal house, descended in the first instance from one or other of the Macedonian marshals to whom Alexander had entrusted the government of provinces. The one which most concerns us here is Egypt, ruled over by the descendants of the general Ptolemaios, in their capital Alexandria, where the great conqueror's body was buried. It was here that the " Museum " was founded under royal patronage, very soon after the accession of the first Ptolemy. It was hardly a museum in the usual sense; being designed rather for a universal library, together with the necessary accommodation and appliances for the use of the scholars who were to work upon this material. It aimed at taking all knowledge for its province; in particular it was to compete with Athens, or rather to replace her. The emissaries of the Ptolemies ransacked Greece for written records of all kinds, and the Museum library was enriched therewith, sometimes by the method of borrowing without the subsequent formality of return. In this way, and by the devoted mass-production of its own indefatigable inmates, it succeeded in amassing about

half a million volumes. This is not as bad as it sounds : for the *Elements of Euclid*, not the least distinguished of this company, would probably count as thirteen, not merely as one volume.

If the social and political background can hardly be called healthy or stimulating, there is still something to be said for the advantages bestowed by the situation on these erudite pets of the Ptolemies. The earlier investigators probably had large public calls made upon their time ; every citizen had been supposed to function as such in reality, according to the old democratic standard, which was not a light one ; and a distinguished and intelligent man would find these calls still heavier. It is certain, for example, that Thales was an active and respected functionary at Miletus. Nor would the Athenian atmosphere be likely to leave such a man as Plato entirely free ; one had, for instance, to spend a large part of one's leisure empanelled in monstrous juries, which were never short of business ; to say nothing of a variety of other public services which could only be avoided by finding a substitute. The Museum worker was freed from all this ; the public services were very efficiently managed by a paid bureaucracy ; and municipal meetings were merely *pro forma.* The happy *savant* could therefore look forward to unclouded years of heuresis, or research, until such time as he became himself an exhibit in the Museum cemetery.

Their labours are not to be despised. In the first place, they saved much of the literature of the previous age, which was otherwise in danger of dissipation and loss ; it contains some really quite good things. Next, they produced a mass of their own to supplement it, on up-to-date scientific principles ; some of this has also survived. It is generally thought by competent critics to be not quite up to the standard of what they retrieved from the previous age ; accordingly, it is not generally taught in schools. In this department atmosphere is wont to exercise a more direct and baneful effect. They invented textual criticism ; this is a cold-blooded science in which the emotions have little part, though I have occasionally heard an emendation described as beautiful ; I have even shed a tear over some of my own.

They went on heartily with the recognised Aristotelian body of natural sciences. In medicine they are allowed by critics to have produced a " rigorously scientific and fundamental literature ", based on the labours of such anatomical investigators as Herophilos ; and so to have worthily followed in the footsteps of Hippocrates ; these results, however, were almost totally lost owing to the traditionalism which overtook medicine in later antiquity.

It was otherwise with their mathematics, of which at least the results were handed on, and indeed became the basis of the renewed mathematical studies at the time of the Renaissance in Europe. Euclid was one of the earliest of the Alexandrian mathematicians ; a star, not of the first magnitude, but one whose methodical virtues have secured immortality where greater lights than he have left little more than a name. For several centuries more, during the gradual decline of the Ptolemaic kingdom in Egypt, mathematical research went on with steady, if not sensational, progress. Out of many names might be mentioned that of Archimedes, who died at Syracuse in Sicily towards the end of the third century. He interested himself in all branches of mathematics, including astronomy ; he seems to have had a special bent towards practical mechanics, in which he is credited with more than one discovery ; he is, I suppose, to be regarded in some degree as the father of that science as well as of hydrostatics. Though much of his work was done in Sicily (where he is said to have met his death at the hands of a Roman soldier whilst studying a mathematical problem), he was probably by intellectual origin an Alexandrian, as it is recorded that his early years of study were spent there. The name of the astronomer Eratosthenes, who incidentally held the position of director of the Museum, also appeals to the layman ; for he achieved the first

approximately correct measurement of the earth's circumference by strictly scientific means, making it 31,000 miles or so, which was not too bad. I realise, however, that this is among the *puerilia* of mathematics.

I will not dwell further upon the Alexandrians ; partly because I feel that your exemplary patience must be almost exhausted ; also because with them, whatever their technical progress may have been, we seem to pass beyond the age of heroes. Yet they handed on the torch to some purpose ; and it is most unlikely that without their aid we should have heard of those very great foundations upon which they themselves built—just as it is most unlikely that they would have started the business for themselves with anything like the same prospect of success. Without their labours, those figures might have come down to us merely as faint and legendary traditions, or encrusted with myth even more thickly than Pythagoras, or than the god Glaucus with his sea-shells. We might well have been disposed to dismiss their scientific achievements as myths too, and to believe that we had discovered everything for the first time. Thanks to Alexandria, however, we know that they were real flesh and blood, and real minds ; and that, too, among the greatest which the surge of protoplasm on this planet has ever thrown up.

It is related of Diogenes, lover of wisdom, a man who had to listen to many readings of many papers, that on one occasion, when he observed in the lecturer's hand a partially written column after many that had been all too densely filled, he turned to his neighbour and remarked : " Courage, my friend ; I see land ". The present page is also partially written, though you cannot see it ; I therefore convey to you the grateful news that you have reached a temporary haven of rest.   W. M. EDWARDS.

GLEANINGS FAR AND NEAR.

814. The Inspector of Schools will not give in. [He sends] letter after letter [to the] *Times* to prove that the earth does not turn on her axis. . . . If a man wished to walk round the circle, and yet keep his nose turned towards a very distant object—say he wanted, without leaving the circle to keep on inspecting a school three miles off—he would be sensible of the effort requisite . . . and would imagine that he had to make a new kind of rotation, whereas all he has to do is to remember not to make the old one. . . . Suppose a person to neglect the gradual turning on the axis until the necessity for it mounts up. When a point travels over the four sides of a square, it moves *round* the centre of the square, though not always at the same distance. Now let a man walk round the square. When he comes to the corner he must make a quarter face, unless he prefer to walk sideways. And this he does four times. [Similarly with an octagon, etc., and] the turns become severally less and more frequent. Finally, at the limit, . . . the figure becomes a circle, the turning becomes gradual, and the successive rectilinear motions merge in a continuous circular motion. If our readers will ponder this explanation a little, they will probably arrive at the conclusion that a person who cannot make it out is not fit to be an inspector of schools.—*Athenaeum*, 1856, p. 492. [Cf. De Morgan, *Budget of Paradoxes*, vol. ii. pp. 4, 5, 20, 84-5 (1915).]

815. The point is an old one. It arises from time to time, and is put down in the opinion of all except those who are put down. Then come Harvey and Jenner, Copernicus and Galileo, and the thing goes to rest. We abide by our opinion that a nose cannot first point north and then south without a right-about face.—Review of *The Moon's Rotation Considered*, by J. L. MacElshender, 1857. *Athenaeum*, 1856, p. 682. [Cf. *Budget of Paradoxes*, 1915, p. 87.]

THREE SADLEIRIAN PROFESSORS: A. R. FORSYTH, E. W. HOBSON AND G. H. HARDY.

By Prof. H. T. H. Piaggio, M.A., D.Sc.

In the present month Professor G. H. Hardy, who for eleven years has been resident in Oxford, will return to Cambridge and occupy the Sadleirian chair of mathematics, vacant by the resignation of Professor Hobson. The occasion seems a favourable one to recall the history of the Sadleirian chair and of the three distinguished living mathematicians who have occupied it. All of them have been Presidents of the Mathematical Association, and have exercised a great influence upon the teaching of our subject. We shall deal at some length with this aspect of their work. Some account will also be given of their researches, but although from a higher standpoint this is the most important of their activities, an adequate estimate of their original work will not be attempted on this occasion.

The foundation of the Sadleirian (or Sadlerian) chair can be traced back [*] to a benefaction of Lady Mary Sadleir, who by her will of 1701 left the University of Cambridge an estate, the income from which was to be used to maintain lecturers in algebra at nine colleges. The benefaction became available in 1710 and the lectureships were duly established. However, with the ever-growing range of studies, the restriction to a single branch of mathematics gradually deprived the lectures of much of their value. In fact, after a time, it was difficult to persuade anyone to attend, and the benefaction ceased to fulfil the wishes of the founder. At last this state of affairs was so obviously bad that a proposal was made to suppress the lectureships and use the money for the endowment of a professorship, to be called the Sadlerian Professorship of Pure Mathematics. This was sanctioned in 1857 and came into operation in 1863. The spelling *Sadlerian*, which occurs in the statute establishing the chair, was always used by its first two occupants, but since their time the form *Sadleirian* has been preferred, as conforming more closely to the name of the founder. The duties of the Professor were to give one course of lectures in one term of the year, and " to explain the principles of pure mathematics." After 1886 the stipend of the post, at first modest, was increased, and two courses of lectures were required. It was hoped that the Sadleirian and other professors of mathematics would come into touch with undergraduate students, but the cast-iron régime of the Mathematical Tripos prevented this. It was impossible for undergraduates, whose future career depended upon their positions in the order of merit in a highly competitive examination, rigidly confined to a stereotyped syllabus, to " waste their time " with professors who were eagerly extending the bounds of knowledge, and seeking after new truths generally too complicated to be dealt with in a three hours' examination. Thus arose the strange paradox that Cambridge possessed a number of eminent professors whose lectures had little (if any) influence on even the best students, and with whom most of the undergraduates were wholly unacquainted.

The first Sadleirian Professor was Arthur Cayley (1821-1895), one of the greatest mathematicians of the nineteenth century. There is no need to go into details of his life and work. An exhaustive account will be found in the article by Professor Forsyth mentioned in the footnote.

The second Sadleirian Professor was Andrew Russell Forsyth. Born in Glasgow on 18th June, 1858, he was educated at Liverpool College and at Trinity College, Cambridge. He graduated as Senior Wrangler and First Smith's Prizeman in 1881 and was elected a Fellow of Trinity in the same year.

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[*] This account of the history of the Sadleirian chair is derived from Professor Forsyth's obituary notice of its first occupant, Professor Cayley (*Proc. Royal Soc.*, Vol. 58, 1895; reprinted in Cayley's *Collected Mathematical Papers*).

For a short time (1882-3) he was professor of mathematics at University College, Liverpool (now Liverpool University). He returned to Cambridge as a College and University lecturer and assistant tutor in 1884. In 1895 he succeeded Cayley, and his first task was to edit the unpublished portion of his predecessor's collected works. He remained as Sadleirian Professor until his resignation in 1910. A brief period spent in India led to the publication in 1913 of his lectures on *Functions of two or more Complex Variables*, delivered, by special invitation, to the professors and doctors of the University of Calcutta. In 1913 he became Chief Professor of Mathematics in the Imperial College of Science and Technology. His retirement from this position in 1923 has not meant a relaxation of his activities, but rather an opportunity for a widening of interests.

Professor Forsyth has always been a prolific writer. He soon acquired a great reputation by his research work and books, which at first were chiefly related to differential equations. His *Treatise on Differential Equations*, first published in 1885 and now in its sixth edition, was described by the *Mathematical Gazette* (II, p. 295, May 1903) as the most lucid, accurate and exhaustive exposition of the subject in our language. It has been translated into both German and Italian. This was followed by his monumental *Theory of Differential Equations*, whose six volumes appeared at intervals from 1890-1906. It is doubtful if even the Germans have produced any treatment of the subject on so extensive a scale. Among Professor Forsyth's own researches Cajori's *History of Mathematics* specially mentions Differential Invariants and Reciprocants, and Singular Solutions. He has applied the methods of differential equations to find systems of invariants and covariants that are algebraically complete, and given a complete discussion of certain differential equations that had been rather cursorily dealt with by writers on relativity. A valuable summary of the present state of knowledge in partial differential equations, indicating opportunities for further research, was given to the London Mathematical Society (Presidential Address) in 1906. But Professor Forsyth's interests have never been restricted to a single subject. In 1893 appeared his *Treatise on the Theory of Functions of a Complex Variable*, now in its third edition. Among the topics on which he lectured during his tenure of the Sadleirian chair may be mentioned Differential Geometry and the Calculus of Variations. His lectures on Differential Geometry appeared in a volume published in 1912. His lectures on the Calculus of Variations were the earliest in Cambridge to expound the Weierstrass theory : these were embodied in a treatise published in 1927 which extended the whole range of the subject and included much new research. In 1928 he edited the late Professor Burnside's *Theory of Probability*, and in 1930 he published his own two-volume *Geometry of Four Dimensions*. Among his minor writings may be mentioned *Mathematics in Life and Thought* (1929) and several biographical notices ; his wide and detailed knowledge enables him to deal with the life and work of eminent mathematicians in a manner impossible to those of more restricted range. Naturally Professor Forsyth has been the recipient of numerous honours. He was elected a Fellow of the Royal Society in 1886, served on the Council 1893-5, and was awarded the Royal Medal in 1897. He was President of the Mathematical Association 1903-5 and of the London Mathematical Society 1904-6. Honorary degrees have been conferred on him by Aberdeen, Calcutta, Christiania, Dublin, Glasgow, Liverpool, Oxford, and Victoria. He is an honorary member of several learned societies, including some in Italy, Russia, and the United States.

The aspect of Professor Forsyth's work that will appeal most to us is the part he played in the improvement of geometrical teaching. For over thirty years the mathematical teachers represented by our Association endeavoured to free the schools from the tyranny of Euclid, but their efforts were in vain until they found allies in the British Association, in engineers like Professor

Perry, and in Cambridge mathematicians like Professor Forsyth. Following a discussion at Glasgow in 1901, in which Professor Perry took the leading part, the British Association appointed a committee to report on improvements that might be effected in the teaching of mathematics. The report of this committee, drawn up by its chairman, Professor Forsyth, will be found in the *Mathematical Gazette* (II, pp. 197-201, October 1902). It was cautiously worded, avoiding the exaggerations that had weakened the arguments of some of the more enthusiastic reformers, and it undoubtedly paved the way for the decisive step, namely the adoption by Cambridge of the recommendations of a Special Syndicate (of which, among others, Professor Forsyth, Messrs. Barnard, Godfrey, and Siddons were members) of a new syllabus in Geometry that for the first time did not make Euclid compulsory. These recommendations related to the Previous Examination: shortly afterwards, they were adopted by the Cambridge Local Examinations Syndicate for their examinations open to the whole country. Thus we may look upon Professor Forsyth as the Moses of our Association, bringing us at last to the promised land of geometrical reform after many weary years in the wilderness. Other contributions to the work of the Association are the opening of the discussion on the *Coordination of the Teaching of Mathematics and Science* (V, pp. 244-252, March 1910), the presidential address to the London branch on *Differential Equations in Mechanics and Physics* (XI, pp. 73-81, May 1922), and the article on *Dimensions in Geometry* (XV, pp. 325-338, March 1931).

The third Sadleirian Professor was Ernest William Hobson. Born at Derby on 27th October, 1856, he was educated at Derby School and Christ's College, Cambridge. He graduated as Senior Wrangler in 1878. He was elected to a Fellowship at Christ's College and became a tutor. In 1903 he became Stokes Lecturer, and he held this post until his election to the Sadleirian chair in 1910. He retained this chair until 30th September of the present year.

In 1891 he published the first edition of his *Treatise on Plane Trigonometry*. The later portions of this book were for many years the only place (with the exception of Chrystal's *Algebra*) where could be found an accurate account in English of complex numbers and of infinite series. In 1907 the fame of his Trigonometry was eclipsed by that of his *Treatise on the Functions of a Real Variable and the Theory of Fourier's Series*. Of this Professor W. H. Young remarks (*Mathematical Gazette*, XI, p. 428, December 1923) that it " was at the time the only systematic account of theories so novel in their character, even to the ordinary professional mathematician, that author and publishers alike may well have had doubts as to the success of the venture." Later on the book was doubled in size and divided into two volumes, which appeared in 1921 and 1926 respectively. A third edition of the first part, still further enlarged, appeared in 1927. The complete work constitutes the most exhaustive account of the subject that has yet appeared in any language. A comprehensive treatise on *The Theory of Spherical and Ellipsoidal Harmonics* is on the point of publication. Professor Hobson's smaller books are *Squaring the Circle* (1913) and *The Domain of Natural Science* (1923 ; a series of Gifford lectures delivered at Aberdeen).

Most of Professor Hobson's researches have been connected with the theory of functions of real variables, but he has also dealt with Legendre's and Bessel's functions, integral equations, potential theory, the conduction of heat, and the calculus of variations. His presidential address to the London Mathematical Society in 1902 was entitled *The Infinite and Infinitesimal in Mathematical Analysis*. The London Mathematical Society's *Proceedings* contain thirty-nine of his papers.

Professor Hobson was elected a Fellow of the Royal Society in 1893, served on the Council 1903-5 and 1928-30, and was awarded the Royal Medal in 1907. The London Mathematical Society chose him as President in 1900-2, and gave

him their De Morgan Medal in 1920. Honorary degrees have been conferred upon him by Aberdeen, Dublin, Manchester, Oxford, St. Andrews, and Sheffield, and he is a member of learned societies in Ireland, Germany, and Italy.

Professor Hobson acted as our President in 1911-3. His presidential addresses were entitled *The Democratization of Mathematical Education*, and *On Geometrical Constructions by Means of the Compass*. These will be found in the *Mathematical Gazette*, VI, pp. 234-242, March 1912, and VII, pp. 49-54, March 1913. Perhaps the greatest service rendered by Professor Hobson to the cause of reform in mathematical teaching was the prominent part he took (in conjunction with Professors Forsyth, Baker, and Hardy) in advocating the abolition of the order of merit in the Mathematical Tripos. The wonder is that there could endure for so long a system which classified William Thomson (afterwards Lord Kelvin) as second to one who was utterly lacking in originality, but this happened in 1845, and it was not until 1909 that the system came to an end, thanks to a group of determined reformers among whom the three Sadleirian Professors were prominent.

The fourth and present Sadleirian Professor is Godfrey Harold Hardy. Born on 7th February, 1877, he was educated at Winchester and at Trinity College, Cambridge. He was Fourth Wrangler in 1898. In 1900 he took the second part of the Tripos and was placed in the First Division of the First Class. In the same year he was elected to a Fellowship at Trinity. For a time he took pupils in conjunction with the other Smith's Prizeman of 1901, Mr. J. H. (now Sir James) Jeans, one teaching pure mathematics and the other applied. He became a lecturer for Trinity College in 1906 and succeeded Dr. H. F. Baker as Cayley lecturer in 1914. These posts were held until 1919, when he was appointed to the Savilian chair of Geometry in the University of Oxford. He resigned this to take up his duties in Cambridge as Sadleirian Professor in October.

Professor Hardy's output of research has been very great. In the London Mathematical Society's *Proceedings* alone there have appeared over sixty papers, and there are a great many more in other English and foreign journals. Most of these deal with convergence of series or the analytic theory of numbers. Several have been written in collaboration with Professor Littlewood. Landau's *Vorlesungen über Zahlentheorie* (1927) gives great prominence to a set of theorems which he calls the first, second, third and fourth Hardy-Littlewood theorems. He also mentions the Hardy identity, the Hardy-Landau identity, and Hardy's theorem on the roots of the Zeta-functions. Hardy's convergence theorem is now standard ; it will be found in Whittaker and Watson's *Modern Analysis*, Chap. VIII. It is interesting to notice that some foreign writers (*e.g.* in *Acta Mathematica*) make the Hardy-Littlewood methods the starting-point of their own work. Many eminent Cambridge mathematicians remain almost unknown to the rest of the mathematical world, but Professor Hardy has never been isolated.

It was one of his writings (the tract on *Orders of Infinity*, 1910) that helped to fire the imagination of the Indian genius Ramanujan. The subsequent correspondence led ultimately to Ramanujan settling in Cambridge. However, he was peculiarly weak in the power of expressing himself, and his papers would probably never have been published if it had not been for the self-sacrificing labours of Professor Hardy in putting them into intelligible form.

Professor Hardy has written three of the Cambridge *Tracts in Mathematics and Mathematical Physics*. One has already been mentioned ; the others are *The Integration of Functions of a Single Variable* (1905) and *The General Theory of Dirichlet's Series* (1915, in collaboration with M. Riesz). His only text-book, *A Course of Pure Mathematics*, first appeared in 1908. The late Mr. A. Berry, in reviewing the fifth edition (*Mathematical Gazette*, XIV, pp. 428-9, April 1929), said " he has shown in this book and elsewhere a power of being interesting, which is to my mind unequalled by any of the eminent

men (with the possible exception of M. Picard) whom I have just mentioned. I suggest to Professor Hardy that he would probably be increasing his service to English mathematics if he were to divert to this purpose " (the writing of a substantial treatise on analysis) " some of the mental energy and time that he would otherwise devote to drawing somewhat closer the *cordon* that surrounds the unknown zeroes of Riemann's Zeta-function, and to similar problems."

Professor Hardy was elected a Fellow of the Royal Society in 1910 and awarded the Royal Medal in 1920. He was President of the London Mathematical Society in 1926-8 and of the Mathematical Association in 1924-6. Honorary degrees have been conferred upon him by Birmingham, Manchester, Marburg, and Oslo, and he is a member of learned societies in Austria, Czechoslovakia, Denmark, Germany, India, Poland, Russia, Sweden, and the United States.

Professor Hardy's contributions to the *Mathematical Gazette* have one dominant theme running throughout. All his life he has been fighting against the tendency for English mathematics to become stereotyped and out of touch with current tendencies abroad. It will be remembered that the generation that followed Newton adhered to his methods exclusively to the neglect of the more powerful methods that had developed upon the continent. This brought Cambridge and English mathematics into an isolated state, from which they were rescued at the beginning of the nineteenth century by the labours of Woodhouse and of the Analytical Society (Peacock, Babbage, and Herschel). Apparently this state of affairs tends to recur. Professor Hardy's earlier contributions to the *Gazette* included several reviews, in which he severely attacked text-books which reproduced the errors which have unfortunately become traditional among English authors. The best statement of his views are contained in his two presidential addresses *What is Geometry ?* (XII, pp. 309-316, March 1925) and *The Case against the Mathematical Tripos* (XIII, pp. 61-71, March 1926). He declared it broadly true that " Tripos mathematics was a collection of elaborate futilities," and quoted the opinion of a foreign friend that the peculiar characteristics of English mathematics had been " occasional flashes of insight, isolated achievements sufficient to show that the ability is really there, but, for the most part, amateurism, ignorance, incompetence, and triviality." These evils he ascribes to the Mathematical Tripos. " The system is vicious in principle, and . . . the vice is too radical for what is usually called reform. I do not want to reform the Tripos, but to destroy it." This address was one of the most striking ever delivered to our Association, and it made a deep impression upon all who listened to it. The majority agreed with the denunciation of the present system, but there was some fear that the drastic remedies proposed might bring forth even worse results.

The Mathematical Association will wish Professor Hardy every success in his tenure of the Sadleirian chair.

H. T. H. PIAGGIO.

UNIVERSITY COLLEGE, NOTTINGHAM.

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816. " ' Since I discovered, several years ago, that I was living in a world where nothing bears out in practice what it promises incipiently, I have troubled myself very little about theories. . . . Where development according to perfect reason is limited to the narrow region of pure mathematics, I am content with tentativeness from day to day.' "—*The Early Life of Thomas Hardy*, ch. xii., p. 201. [Per Mr. P. J. Harris.]

817. " ' Have been thinking over the dictum of Hegel. . . . These venerable philosophers seem to start wrong . . . ; it was Comte who said that ' metaphysics was a mere sorry attempt to reconcile theology and physics.' "—*The Early Life of Thomas Hardy*, ch. xiv., p. 234. [Per Mr. P. J. Harris.]

SOME POINTS IN THE TEACHING OF PURE GEOMETRY.

By H. G. Green, M.A., University College, Nottingham.

The discussions which have taken place on the methods of teaching Pure Geometry have been concerned almost entirely with the elementary stages, and the problem of the more advanced teaching has hardly been touched. We should, however, face the phenomenon which occurs time after time—of the boy of undoubted mathematical ability, proceeding to specialise in mathematics, who makes good progress in other branches but who fails to make any real headway in Pure Geometry. There is no doubt that difficulties in the earliest stages, however ably overcome, must of necessity foreshadow further difficulties in the more advanced ones. We propose, therefore, to touch on some points in the senior course. We limit ourselves to the discussion of general points and omit difficulties of detail that arise in the teaching of particular theorems, though we are aware that the problems which arise in the teaching of any section of Pure Geometry cannot be fully solved by examining that section alone. Moreover we do not attempt to discuss the philosophical aspect of the subject, but take the viewpoint of the teacher, whose first desire is to present an effective and interesting weapon for mathematical work.

One of the greatest dangers which the teacher of Pure Geometry has to face is that of giving the impression that the subject consists of a large number of independent tricks, and that to be successful the student must develop an uncanny memory and a knack of sorting out, without any obvious reason, the correct trick for the question in hand. This danger can largely be met by a recognition of a major grouping. In the one suggested the groups work in pairs, in which the first of the pair (odd numbered groups) really prepares the mechanism for the general processes of the second (even numbered groups). The existence of a trend of policy through the pairs should be duly emphasised in the course of teaching and should be used to bind together the conceptions developed in the second group of the pair. Group 1 in the scheme presented below represents the elementary stage which occurs before our range is entered upon ; group 2 is a borderline group which is sometimes covered by non-specialists desirous of a little wider knowledge.

GROUPS

1. Properties of the straight line and circle as represented by the ordinary pre-specialist course of geometry based on Euclid and the revised systems of elementary geometry based on Euclid.
2. Systems of straight lines and circles represented for the straight line, by rather more advanced geometry of the triangle and, for the circle, by the coaxal system, and leading in conclusion to the process of inversion for which the triangle and coaxal properties serve as primary examples.
3. Properties of the conic sections (geometrical conics).
4. Systems of conics handled by reciprocation.
5. Projective geometry of anharmonic and harmonic ranges and pencils ; involution ; projection.
6. Systems of conics and their properties as derived in group 5 handled by generalised projection.

The order in which the various branches of Pure Geometry should be taught is a matter of great difficulty. The order given in the grouping of the previous paragraph indicates a logical development, but it is sometimes convenient to deal with a subject earlier than its logical position justifies so as to avoid some lengthy detour, and thus, for example, the treatment of harmonic ranges, either in full or to a limited extent, may be taken from the fifth group and placed in the second. There is also, in addition to this question of convenience, a

certain amount of freedom for some of the parts, and the retention of the logical order may become a matter of pedantry disadvantageous to practical teaching ; for these the order should be broken to conform to the moods of the class.   For example, with a class with a spirit of adventure the work on reciprocation with the rather dreary work on conic properties which leads up to it, and the still more limited half section on inversion, may well be postponed in order that the introduction to the more novel and wider processes of projection may be hastened.   This does not imply a breakdown of the sectional scheme, as groups 3, 4 would still be kept together, and group 2 still has the properties of a system as its climax, and the connection of inversion with this group can be made in a few words which in themselves give a preliminary interest to the subject.

The refusal to admit analytical processes may prove a severe handicap to the progress of the student.   This has already been recognised to a considerable extent in dealing with the properties of the conic section (group 3), where the methods of Pure Geometry are best used to supplement those of Algebraic Geometry.   In involution, and in (1,1) correspondences, the use of algebraic equations not only provides summaries of previous work but also gives convenient bases for further developments.   Again, in dealing with projections, as soon as the conception of the " line at infinity " is needed the analytical process is almost essential if the young student is to obtain any coherence in his ideas.   Indeed, whatever may be the attitude of the philosopher, there is at this stage at any rate no fundamental difference between Pure and Algebraic Geometry ; the only difference between them lies in the fact that they adopt different languages to describe the same thing, the position of a point, but the basic ideas are essentially the same.

We do not discuss in this article the difficulties of detail involved in the actual teaching of a particular theorem, as they are of the same general type as those involved in the teaching of the more elementary stages.   There is, however, one point which appears to be of great importance.   At this stage we are dealing with students who, for the greater part, are entering upon a course of specialist mathematics, and we are therefore apt to think of them as having somewhat of the point of view of the polished mathematician.   We present to them the proof of some, to us, elegant property and expect them to hold it in the same regard.   It is almost as ambitious to expect them to appreciate the final step of a group of processes as it was, in the old days, to expect a small boy to appreciate Pythagoras' Theorem as the " point culminant " (except in the literal sense with a sigh of relief at reaching the end of it) of his first introduction to Euclid.   To meet this difficulty of lack of quick appreciation, the writer suggests that after the introduction and proof of every new property the teacher should firmly link with it all outstanding deductions and dependent properties.   Moreover, when a related sub-group has been dealt with in this way it should be followed immediately by a short series of riders illustrative of all parts of the sub-group until the basic figure and its properties have made a definite impression on the student.   This impression, once thoroughly made, unconsciously reacts on the mind of the student to lead him finally to a true appreciation of the value and elegance of the central property.

The final plea which the writer wishes to make is that a powerful weapon in the teaching of Pure Geometry is often thrown away by the neglect of practical drawing exercises.   To take a concrete example, we have early in the theory of involution ranges a theorem regarding the requisite number of pairs of points necessary to determine a range.   This is followed time and time again by such statements as " let X' be the mate of the point X " or " these collinear points belong to an involution range since they lie also on a certain group of lines ", implying the results of a geometrical construction which has only been tacitly implied in some other theoretical process.   Again,

we have an almost endless series of such statements as " draw the conic through the five points ", or " let the conics of the four-point system which touch a line, etc. . . ." An accumulation of such steps, while quite sound logically, is very apt to produce an atmosphere of vagueness quite parallel to one which would be produced in algebra if the roots of an equation were always discussed (and in fact the simile is at times more than superficial) without any equations ever being formed. The writer does not wish to suggest that every construction which occurs in the work should be actually drawn in full detail any more than that it is necessary that every equation must be formed in full before its roots can be discussed, but he does feel that, even in the more advanced stages of the work, the practical performance of a few well-chosen constructions will give confidence and interest—full recompense for the time thus employed.   H. G. GREEN.

818. We may doubt the warranty of the priest, but never that of the mathematician.—H. M. Tomlinson, *All Our Yesterdays*, p. 10.

819. Every man who is not a monster, a mathematician, or a mad philosopher, is the slave of some woman or other.—*The Sad Fortunes of the Rev. Amos Barton*, chap. iv. [Per Mr. James Buchanan.]

820. (Description of Kharama.) It was the thin, high-boned, high-bred face of the hillman. . . . The brow was straight and heavy, such as I had always associated with mathematical talent.—John Buchan, *The Three Hostages*, chap. ix. [Per Mr. James Buchanan.]

821. " Il s'informa s'il y avait des prisons, et on lui dit que non. Ce qui le surprit davantage, et qui lui fit le plus de plaisir, ce fut le palais des sciences, dans lequel il vit une galerie de deux mille pas, toute pleine d'instrumens de mathématique et de physique."—Voltaire, *Candide*, ch. xviii. [Per Mr. J. B. Bretherton.]

822. " I asked him whether he thought the tricks it was now considered cultured to play with mathematics came within the category of this intellectual decay. The old gentleman answered me a little abruptly that he could not judge what I was talking about.

'Why,' said I, ' do you believe that parallel straight lines *converge* or *diverge* ? '

' Neither,' said he, a little bewildered. ' If they are parallel they cannot by definition either diverge or converge.'

' You are, then,' said I, ' an old-fashioned adherent of the theory of the parabolic universe ? ' At which sensible reply of mine the old man muttered rather ill-temperedly, and begged me to speak of something else."—Hilaire Belloc, *The Old Gentleman's Opinions* (in his collection of essays, *First and Last*). [Per Mr. H. A. Hayden.]

823.   BARROW AND NEWTON.

Barrow. Speak it out, man ! Are you in a ship of Marcellus under the mirror of Archimedes, that you fume and redden so ? Cry to him that you are his scholar, and went out only to parley.

Newton. Sir ! in a word—ought a studious man to think of matrimony ?

Barrow. Painters, poets, mathematicians, never ought . . . other studious men, after reflecting for twenty years upon it, may . . .

Newton. Supposing me no mathematician, I must then reflect for twenty years !

Barrow. Begin to reflect on it after the twenty ; and continue to reflect on it all the remainder ; I mean at intervals, and quite leisurely. It will save to you many prayers, and may suggest to you one thanksgiving.

—W. S. Landor, *Imaginary Conversations.*

MATHEMATICAL NOTES.

1005. [P. 3. b.] *A common error in the Theory of Inversion.*

§ 1. Casey in his *Sequel* states that

(1) *If two circles be inverted into two others, the square of the common tangent of the first pair divided by the rectangle contained by their diameters is equal to the square of the common tangent of the second pair divided by the rectangle contained by their diameters.*

The statement as it stands is meaningless, as it is not said whether the common tangent, if existent, is to be direct or transverse. Most readers naturally assume that it is intended that the direct common tangent should be taken for the inverse circles if the direct common tangent was taken for the given circles, and the transverse, if the transverse. This is what Coolidge does, *Circle and Sphere*, p. 37. My object is to point out that *the proposition, so understood, is not true.*

Casey also states (I have added some conditions he obviously assumes) :

(2) *If two given circles are cut by their line of centres, one in A, B, the other in C, D, and B, C be between A and D, and if the inverse circles of these be cut by their line of centres, one in A′, B′, the other in C′, D′, and B′, C′ lie between A′ and D′, then*

$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}. \quad \dots\dots\dots\dots\dots\dots\dots\dots(i)$$

This is practically the statement in Lachlan, *Pure Geometry* (1893), p. 227. *It is also not true.*

§ 2. Of the following, (1) and (2) are easily proved :

(1) *If one circle be inverted into another, the centre of inversion is inside both or outside both, and is between the centres of the circles in the first case and not between them in the second case.*

(2) *If A be the centre of the given circle, B that of its inverse, and if r, ρ be their radii, and O, k the centre and radius of inversion, then*

$$OA : OB = r : \rho = \pi : k^2, \quad \dots\dots\dots\dots\dots\dots\dots\dots(ii)$$

where $\pi = OA^2 - r^2$ *or* $r^2 - OA^2$, *according as O is outside or inside the given circle.*

(3) *Consider now two circles with radii r_1, r_2, centres A_1, A_2 at a distance d apart, and let them be inverted into two circles with radii ρ_1, ρ_2 and centres B_1, B_2 at a distance δ apart. Let O, k be as before, then*

$$\frac{d^2 - r_1^2 - r_2^2}{2 r_1 r_2} = \pm \frac{\delta^2 - \rho_1^2 - \rho_2^2}{2 \rho_1 \rho_2}. \quad \dots\dots\dots\dots\dots\dots(iii)$$

For we have

$$B_1 B_2{}^2 = B_1 O^2 + B_2 O^2 - 2 B_1 O \cdot B_2 O \cos B_1 O B_2,$$
$$\rho_1 \rho_2 = k^4 r_1 r_2 / \pi_1 \pi_2, \text{ by (ii) ;}$$

where $\pi_1 = OA_1{}^2 - r_1{}^2$ or $r_1{}^2 - OA_1{}^2$ according as O is outside or inside the circle centre A_1, and π_2 has a similar meaning for the circle, centre A_2.

We must consider the various cases :

(a) Suppose O outside both given circles, then the angles $A_1 O A_2$, $B_1 O B_2$ are equal. An easy calculation gives (iii) with the *positive* sign. (Details will be given in the *Higher Course Geometry* now in the press.)

(b) The same result follows if O is inside both given circles.

(c) But if O is inside one and outside the other given circle, we obtain (iii) with the *negative* sign.

§ 3. The lengths of the direct and transverse common tangents of the circles A_1, A_2, if these tangents exist, are given by

$$T^2 = A_1 A_2{}^2 - (r_1 - r_2)^2, \quad t^2 = A_1 A_2{}^2 - (r_1 + r_2)^2.$$

We shall use these expressions to define T^2 and t^2, even if these are negative, and the tangents therefore nonexistent. The corresponding magnitudes for the circles, centres B_1, B_2 will be denoted by T'^2 and t'^2.

Now adding unity to (iii) we have at once

$$T^2/r_1r_2 = T'^2/\rho_1\rho_2,$$

provided the sign of the R.H.S. of (iii) is positive, *i.e. provided the centre of inversion is outside both the given circles.* Similarly, under these conditions,

$$t^2/r_1r_2 = t'^2/\rho_1\rho_2.$$

But if the centre of inversion is inside one and outside the other given circle we must take the negative sign in (iii), *and then we obtain*

$$T^2/r_1r_2 = -t'^2/\rho_1\rho_2, \quad t^2/r_1r_2 = -T'^2/\rho_1\rho_2.$$

This is given correctly in Lachlan, p. 232, and incorrectly, or at least with the conditions omitted, on p. 228. The formula at the top of Coolidge, p. 37, is also unsound.

The case last mentioned can readily occur, and the incorrect formulae lead to wrong results in the real domain. For consider two circles which cut, for which therefore T^2 is positive. Invert them for a point inside one and outside the other. T'^2 is still positive, but T^2/r_1r_2 is not equal to $T'^2/\rho_1\rho_2$ but to $-t'^2/\rho_1\rho_2$, although the transverse common tangent does not exist.

§ 4. If the points A, B, C, D be arranged as in § 1 (2), then

$$T^2 = AC \cdot BD, \quad t^2 = BC \cdot AD,$$

the intervals AC, etc., being taken with signs. In view of the results in § 3 it is clear that the statement in § 1 (2) is faulty, and the reader will now be able to state it correctly.   H. G. F.
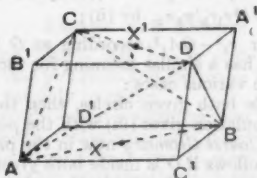
1006. [K¹. 13. c.]   *A Theorem in Coolidge's "Circle and Sphere".*

On p. 235 of his monumental work, Coolidge uses a hyperboloid as an auxiliary for establishing a theorem on the elementary geometry of a tetrahedron, and remarks that though the use of a hyperboloid is "highly unsportsmanlike, frankly the author does not know how to dispense with it in this case".

An examination of the argument seems to show that all that is needed are the properties of the point which is the centre of the hyperboloid that has the altitudes of the tetrahedron for generators. This note considers this point in elementary fashion.

§ 1. Consider the parallelepiped in the figure, and the tetrahedron whose vertices are A, B, C, D, and the tetrahedron with vertices A', B', C', D'. Given any tetrahedron $ABCD$, we can construct a parallelepiped related to it like the one in the figure, and then we can construct the tetrahedron $A'B'C'D'$.

Call such tetrahedra *associate*. The relation between them is obviously a symmetrical one.

Let the diagonals AB, $C'D'$ cut in X, and the diagonals $A'B'$, CD cut in X'.

The plane through X (which is the mid-point of AB), perpendicular to AB, is perpendicular to $A'B'$.

Hence the plane through the mid-point of an edge of $ABCD$ and perpendicular to that edge is a plane through the mid-point of an edge of $A'B'C'D'$ and perpendicular to its opposite edge.

But the planes through the mid-points of the edges of $ABCD$ and perpendicular to those edges cut in the circumcentre O of $ABCD$. Thus the planes through the mid-points of the edges of $A'B'C'D'$, and perpendicular to their *opposite* edges, meet in a point, the circumcentre O of $ABCD$. Similarly the planes through the mid-points of the edges of $ABCD$ perpendicular to their opposite edges, meet in a point, the circumcentre O' of $A'B'C'D'$.

Now the associate tetrahedra have edges equal and parallel in pairs, *e.g.* AB and $A'B'$. The tetrahedra are thus (inversely) congruent. Thence the distance OX of O from AB equals the distance $O'X'$ of O' from $A'B'$.

Also $OX \parallel O'X'$, for the edges of associate tetrahedra are parallel in pairs. Hence XX' and OO' bisect each other.

Similarly the mid-point of OO' is the mid-point of each join of the mid-points of opposite edges of $ABCD$, and hence is the centroid G of $ABCD$ (and of $A'B'C'D'$). Calling O' the associate centre of $ABCD$, we have

The associate centre, which is the cut of planes each through the mid-point of one edge perpendicular to the opposite edge, is such that the centroid is midway between it and the circumcentre.

§ 2. The question is also easily treated by vectors.

Take the circumcentre O for origin, and denote the vectors from O to A, B, ... G by a, b, ... g. Then $a - b$ is the vector from B to A. Denote the scalar product of two vectors a, b by $a \mid b$, and denote $a \mid a$ by a^2. Then

$$a^2 = b^2 = c^2 = d^2. \quad g = \tfrac{1}{4}(a + b + c + d).$$

Take a point P so that the vector $OG =$ vector GP, that is, so that $2g = p$. Now

$$(\tfrac{1}{2}\overline{a + b} - \tfrac{1}{2}\overline{a + b + c + d}) \mid (c - d) = (-\tfrac{1}{2}\overline{c + d}) \mid (c - d) = \tfrac{1}{2}(d^2 - c^2) = 0. \quad \ldots(\text{i})$$

But since $\tfrac{1}{2}(a + b)$ is the vector from O to X, the mid-point of AB, and since $p = \tfrac{1}{2}(a + b + c + d)$, therefore $\tfrac{1}{2}(a + b) - \tfrac{1}{2}(a + b + c + d)$ is the vector from P to X.

Hence (i) means, PX is perpendicular to CD.

Similarly, the line from P to the mid-point of any edge is perpendicular to the opposite edge, whence it easily follows that P is the point where those planes cut which pass through the mid-point of one edge and are perpendicular to the opposite edge; P is therefore the associate centre of $ABCD$.

The corresponding theorem for *cyclic* quadrilaterals in a plane can easily be stated and shown by either of the above two methods.

Continuing the argument, we can derive Coolidge's Theorem 31, p. 237, either by pure geometry as he does, or by vectors, as follows:

A point L one-third of the way from P to D has the vector

$$l = \tfrac{1}{3}(2p + d) = \tfrac{1}{3}(a + b + c + 2d).$$

The centroid G_1 of the face ABC has the vector $g_1 = \tfrac{1}{3}(a + b + c)$.

The point Q one-third of the way from G along GP has the vector

$$q = \frac{1}{3}\left(2 \cdot \frac{a + b + c + d}{4} + \frac{a + b + c + d}{2}\right) = \tfrac{1}{3}(a + b + c + d),$$

thence
$$(q - g_1)^2 = (q - l)^2.$$

Hence the point Q is the centre of a sphere through the centroids of the faces of the tetrahedron and through the points one-third of the way from the associate centre to the vertices.

The rest of Theorem 31 (*loc. cit.*) is easily shown by elementary arguments from this result.   H. G. F.

1007. [D. 5. f.] *The deduction of Saalschütz's theorem from Vandermonde's.*

Dr. Bailey's proof * of Saalschütz's theorem can hardly be regarded as "very elementary". The following modification is, I think, preferable. The method is Thomae's.† We have

$$_3F_2\begin{bmatrix} a, \beta, -n; 1 \\ \delta, \epsilon \end{bmatrix} \equiv \sum_{r=0}^{n} (-)^r {}_nC_r \frac{(a)_r(\beta)_r}{(\delta)_r(\epsilon)_r},$$

and hence, by Vandermonde's theorem,

$$_3F_2 = \sum_{r=0}^{n} (-)^r {}_nC_r \frac{(a)_r}{(\delta)_r} {}_2F_1\begin{bmatrix} \epsilon-\beta, -r; 1 \\ \epsilon \end{bmatrix}$$

$$= \sum_{r=0}^{n} \sum_{t=0}^{r} (-)^{r+t} {}_nC_r {}_rC_t \frac{(a)_r(\epsilon-\beta)_t}{(\delta)_r(\epsilon)_t}.$$

We substitute $t+p$ for r and apply Vandermonde's theorem again, whence

$$_3F_2 = \sum_{t=0}^{n} \sum_{p=0}^{n-t} (-)^p \frac{n!(n-t)!}{(n-t)!\,t!\,(n-t-p)!} \frac{(a)_t(a+t)_p(\epsilon-\beta)_t}{p!(\delta)_t(\delta+t)_p(\epsilon)_t}$$

$$= \sum_{t=0}^{n} {}_nC_t \frac{(a)_t(\epsilon-\beta)_t}{(\delta)_t(\epsilon)_t} {}_2F_1\begin{bmatrix} a+t, t-n; 1 \\ \delta+t \end{bmatrix}$$

$$= \sum_{t=0}^{n} {}_nC_t \frac{(a)_t(\epsilon-\beta)_t}{(\delta)_t(\epsilon)_t} \frac{(\delta-a)_{n-t}}{(\delta+t)_{n-t}},$$

so that

$$_3F_2\begin{bmatrix} a, \beta, -n; 1 \\ \delta, \epsilon \end{bmatrix} = \frac{(\delta-a)_n}{(\delta)_n} {}_3F_2\begin{bmatrix} a, \epsilon-\beta, -n; 1 \\ \epsilon, 1-\delta+a-n \end{bmatrix}.$$

This is Bailey's formula (2).

In Saalschütz's case $\delta+\epsilon-a-\beta+n=1$ and

$$_3F_2\begin{bmatrix} a, \beta, -n; 1 \\ \delta, \epsilon \end{bmatrix} = \frac{(\delta-a)_n}{(\delta)_n} {}_2F_1\begin{bmatrix} a, -n; 1 \\ \epsilon \end{bmatrix}$$

$$= \frac{(\delta-a)_n(\epsilon-a)_n}{(\delta)_n(\epsilon)_n}$$

$$= (-)^n \frac{(\delta-a)_n(\delta-\beta)_n}{(\delta)_n(\epsilon)_n}.$$

F. J. W. W.

1008. [B. 1. d.] *Note on the Kinematics of a System of Three Particles.*

If we refer the moving particles to a system of axes, fixed in direction, passing through their centre of gravity, we have the three equations :

$$m_1x_1 + m_2x_2 + m_3x_3 = 0, \quad m_1y_1 + m_2y_2 + m_3y_3 = 0, \quad m_1z_1 + m_2z_2 + m_3z_3 = 0.$$

Differentiating these, with respect to the time, we have

$$m_1x'_1 + m_2x'_2 + m_3x'_3 = 0, \quad m_1y'_1 + m_2y'_2 + m_3y'_3 = 0, \quad m_1z'_1 + m_2z'_2 + m_3z'_3 = 0;$$

from which we deduce

$$\frac{m_1}{(y'_2z'_3)} = \frac{m_2}{(y'_3z'_1)} = \frac{m_3}{(y'_1z'_2)} = \frac{1}{k_1}, \quad \frac{m_1}{(z'_2x'_3)} = \frac{m_2}{(z'_3x'_1)} = \frac{m_3}{(z'_1x'_3)} = \frac{1}{k_2},$$

$$\frac{m_1}{(x'_2y'_3)} = \frac{m_2}{(x'_3y'_1)} = \frac{m_3}{(x'_1y'_2)} = \frac{1}{k_3}.$$

From these we immediately deduce the equations

$$k_1x'_1 + k_2y'_1 + k_3z'_1 = 0, \quad k_1x'_2 + k_2y'_2 + k_3z'_2 = 0, \quad k_1x'_3 + k_2y'_3 + k_3z'_3 = 0,$$

* *Math. Gazette* 15 (Jan. 1931) 299.   † *Journal für Math.* 87 (1879) 26.

since $(x'_1 y'_2 z'_3) = 0$. Thus the velocities of the three particles, relative to their centre of gravity, are instantaneously parallel to a plane of variable orientation, the direction ratios of whose normal are $k_1 : k_2 : k_3$.

Further, if we write $K^2 = k_1^2 + k_2^2 + k_3^2$, we have

$$m_1^2 K^2 = (y'_2 z'_3)^2 + (z'_2 x'_3)^2 + (x'_2 y'_3)^2 = V_2^2 V_3^2 \sin^2 a,$$

where

$$V_2^2 = x'_2^2 + y'_2^2 + z'_2^2, \quad V_3^2 = x'_3^2 + y'_3^2 + z'_3^2, \quad V_2 V_3 \cos a = x'_2 x'_3 + y'_2 y'_3 + z'_2 z'_3.$$

Thus we get a set of three equations of the type $m_1 K = V_2 V_3 \sin a$, from which we deduce

$$\frac{m_1 V_1}{\sin a} = \frac{m_2 V_2}{\sin \beta} = \frac{m_3 V_3}{\sin \gamma} = \frac{V_1 V_2 V_3}{K},$$

where a, β, γ are the angles of the triangle formed when we project the instantaneous directions of relative velocity on the plane to which they are parallel.   J. Brill.

1009. [D. 2. c.] *Stirling's Theorem.*

Two notes (951, 970) in the *Gazette* during 1930 were concerned with this theorem. A related method of approaching it which arose in quite a different way may perhaps be of interest.

If $a_n = \dfrac{n!}{n^n}$, the ratio test applied to Σa_n gives

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)^{-n} \to \varepsilon^{-1}.$$

But when $\dfrac{a_{n+1}}{a_n}$ tends to a limit, $(a_n)^{\frac{1}{n}}$ has the same limit.

We may therefore write $\dfrac{n!}{n^n} = \lambda_n e^{-n}$ where $(\lambda_n)^{\frac{1}{n}} \to 1$,

$$\text{i.e. } n! = \lambda_n \left(\frac{n}{e}\right)^n. \quad\quad\quad\quad\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If we apply this to Wallis' formula

$$\frac{2^{2n}(n!)^2}{(2n)!} \sim \sqrt{n\pi}, \text{ we obtain } \frac{\lambda_n^2}{\lambda_{2n}} \sim \sqrt{n\pi}. \quad \ldots\ldots\ldots\ldots(2)$$

Further, $\dfrac{\lambda_{n+\kappa+1}}{\lambda_{n+\kappa}} = \dfrac{e}{\left(1 + \dfrac{1}{n+\kappa}\right)^{n+\kappa}}$ by (1),

so that, $\log \dfrac{\lambda_{n+\kappa+1}}{\lambda_{n+\kappa}} = 1 - (n+\kappa)\left\{\dfrac{1}{n+\kappa} - \dfrac{1}{2(n+\kappa)^2} + \ldots\right\}$

$$= \frac{1}{2} \cdot \frac{1}{n+\kappa} + O\left(\frac{1}{n^2}\right) \text{ uniformly in } \kappa.$$

Summing for $\kappa = 1, 2, 3, \ldots n$,

$$\log \frac{\lambda_{2n+1}}{\lambda_{n+1}} = \frac{1}{2}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right\} + O\left(\frac{1}{n}\right)$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n}\right\} + O\left(\frac{1}{n}\right).$$

The right-hand side $\to \frac{1}{2} \log 2$.

Hence both $\dfrac{\lambda_{2n+1}}{\lambda_{n+1}}$ and $\dfrac{\lambda_{2n}}{\lambda_n} \to \sqrt{2}$.

Finally from (2) $\lambda_n \sim \sqrt{2n\pi}$,

and Stirling's result follows.   B. E. Lawrence.

REVIEWS.

The Genetical Theory of Natural Selection. By R. A. FISHER, F.R.S.
Pp. xiv + 272. 17s. 6d. 1930. (Oxford University Press.)

The birth of a new branch of applied mathematics is a rare event, and one which probably never fails to react favourably on pure mathematics within a generation or so. Up till now the vast majority of the problems with which biologists have presented mathematicians have either been trivial or have involved the Gaussian error function and its integral. This was mainly because the biologists were generally concerned to know how much they could say about a population after they had investigated a sample of it, partly because certain quantitative characters are distributed in populations according to Gauss's law. But since the rediscovery of Mendel's laws, which showed that the basis of heredity is atomic, a new era has opened. The new biometry is related to the old much as is statistical mechanics to thermodynamics.

The first five chapters of Dr. Fisher's book deal with some of the mathematical problems raised in the theory of natural selection. Many of them are treated in a rather summary manner, but a fairly full account is given in Chapters IV and V of the problem which the author has done so much to solve. It may be stated as follows : " If the heritable variation of a large population is due to a number of genes which arise by a process of mutation, and tend to be extinguished by random dying out of individuals and sometimes by selection as well, discuss the possible equilibria and the change of the population from generation to generation." The meaning of random extinction can be explained to non-biologist by the following problem. " If m members of a population of N males are called Smith, and the population remains constant, everyone having an equal chance of producing a son, what is the probability of there being s Smiths after n generations ? In particular, what is the chance of the name dying out ? "

If the probability of one man having r sons is the coefficient of t^r in $f(t)$, clearly $f(1) = f'(1) = 1$, and $f(0)$ is the chance of any given man having no sons. The probability of the m Smiths having s sons in all is the coefficient of t^s in $[f(t)]^m$. The same probability in the second generation is the coefficient of t^s in $\{f[f(t)]\}^m$, and so on. If there is some advantage, from the Darwinian point of view, in being called Smith, then $f'(1) > 1$. Fisher considers in detail the case where $f(t) = e^{c(t-1)}$ where c is a quantity equal to or slightly different from unity. If $m = 1$, and u_n the probability that there will be no Smiths after n generations, then $u_{n+1} = f(u_n)$, and $u_0 = 0$. In Fisher's special case $u_{n+1} = e^{u_n - 1}$, an equation which he considers at considerable length, and shows that $u_\infty = 1$, and as $n \to \infty$,

$$n = \frac{2u_n}{1 - u_n} + \tfrac{1}{3} \log_e (1 - u_n) + \cdot 02930.$$

But if $\qquad c = 1 + k, \quad u_\infty = 1 - 2k + \tfrac{8}{3}k^2 \dots$.

Now let us suppose that every individual has a number of hereditary surnames, and we get a nearer approach to the situation considered by Fisher. Further, if a benevolent government creates new surnames at a rate just sufficient to balance random extinction, we have an equilibrium. Among the more important equations arising are

$$\phi(e^{x-1}) - \phi(x) = \tfrac{1}{2} \quad \text{and} \quad \phi(e^{x-1}) - \phi(x) = 1 - x.$$

As a preliminary to solving both, we put $x = u_n$.

This type of method is suitable for dealing with fairly rare " names ", *i.e.* gene differences. For the more frequent ones, if s be the number of persons out of N bearing a given name, $\cos \theta = 1 - \frac{s}{N}$, and y be the frequency of any differential element $d\theta$, he obtains equations such as

$$\frac{\partial y}{\partial t} = -\tfrac{1}{2}k \frac{\partial}{\partial \theta} y \sin \theta + \frac{1}{2N} \frac{\partial}{\partial \theta} y \cot \theta + \frac{1}{2N} \frac{\partial^2 y}{\partial \theta^2},$$

where k is a measure of selective advantage. These equations describe the change or equilibrium of the population under natural selection, but here much remains to be done.

The other problems dealt with are of less mathematical interest, but often of great biological importance. In particular, the fundamental fact is demonstrated that a population evolving under natural selection may acquire, so to say, momentum, and continue to evolve beyond the optimum.

Several points in this part of the book call for criticism. I believe that I have shown that the equilibrium discussed on p. 102 is unstable,* and that the argument of Chapter III breaks down when a certain very large number, tacitly treated as if it were infinite, is realised to be finite. And some readers would have found the book more valuable had it referred to other researches on the same topic. For example, I have published a more general though much less powerful treatment of the " Smith problem " involving Koenigs' theorem on iteration. Finally, a large number of facts bearing on the problem of evolution which are not susceptible of mathematical treatment have been passed over. Such omissions are perfectly justifiable in a book which breaks new ground at innumerable points, and does not purport to be a text-book.

Then follow two chapters dealing with sexual selection and mimicry. Here the argument is logical rather than mathematical, and although it must certainly be taken into account in all future discussions of these topics, I suspect that more observation and experiment are required before any finality is possible. The latter half of the book deals with eugenics. Here again there are no explicit mathematics, but it will be hard reading for anyone who is not fairly familiar with statistical methods. In spite of an emotional attitude regarding contraception and other subjects which the author rightly does not attempt to hide, the argument is on the whole far more objective than is usual when eugenics are discussed. Dr. Fisher believes that a fundamental cause of the higher birth rate among the poor as compared with the rich is as follows.

Besides ability and luck, infertility, which is inherited, is rewarded by wealth and social standing. Members of small families profit both by better education and inherited wealth. Hence the richer classes tend to become congenitally infertile, and those who rise into them by innate ability are partly sterilized by marrying into infertile stocks. If this can be proved conclusively it follows that many of the measures which are alleged to be eugenic would have the opposite effect. Dr. Fisher believes that the dysgenic tendencies of modern society could be abolished by a system of family allowances. To the reviewer his arguments seem to lead more logically to an extreme form of socialism.

However that may be, the book before us should ultimately lead to two desirable results. Discussions of Darwinism by persons wholly ignorant of mathematics will come to be classed with circle-squaring and earth-flattening. And finite difference equations, and in general, functions of an integral variable, will come to their own. For these equations are the natural means by which we connect the parameters of one generation and its successor. Problems involving mating systems give us linear equations. A full discussion of brother-sister mating involves the solution of 55 simultaneous linear difference equations, which may account for the almost universal abhorrence in which this practice is held. As soon as the population is acted on by selection, random extinction, or mutation, the equations cease to be linear. When generations overlap they become integral equations. You cannot discuss the chromosomes, the physical basis of heredity, without bringing in the Macmahon-Hardy-Ramanujan partition function.

Doubtless the day is still distant when " a population in which 17 per cent. of all marriages are between first cousins " will replace the " small weightless elephant " and " perfectly rough insect " of our childhoods. But meanwhile Dr. Fisher's book should serve not only to raise the discussion of the evolution problem to a higher level, but to introduce mathematicians to a new growing point of their subject.   J. B. S. HALDANE.

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* Proc. Camb. Phil. Soc., vol. 27, p. 137 ; 1931.

A Manual of Greek Mathematics. By SIR THOMAS HEATH, K.C.B., K.C.V.O., F.R.S. Pp. xvi + 552. 15s. 1931. (Clarendon Press.)

Whenever a volume by Sir Thomas Heath appears, the mathematical world must feel afresh how deeply indebted it is to the great scholar, who seems so nobly to respond to Cory's invocation, that

" Two minds shall flow together, the English and the Greek ".

Ten years ago the *History of Greek Mathematics* appeared, and at once took its place as the classic account of the progress of mathematics from Thales to Diophantus. There is no need to comment on the merits of the two volumes of the *History* ; in the *Gazette* it received a masterly review by Mr. Greenstreet.* But the price of the *History*, though not exorbitant for two such well-made volumes stored with the learning of a master and written with the love of an enthusiast, was too high to allow the book to appear on the shelves of every mathematician who would have wished to possess it. The *Manual* covers the same ground as the *History*—but the treatment naturally has not the same fullness of detail. That this is so is not due merely to a desire to produce a small volume, but also to a hope that those who are interested in Greek mathematics will be able to refer to this work and read it without having to grind through detail more interesting to the classical scholar or professional mathematician than to the general reader. Anyone who desires to make himself familiar with the amazing progress of mathematical learning in the Greek world should read this fascinating little book. In particular, we would recommend it to all teachers of mathematics who believe that no teacher can be successful unless he has some knowledge of the origins and development of his subject. There may be mathematicians who believe that their acquaintance with the latest text-books on geometry excuses their ignorance of Apollonius, just as there may be classical scholars who hold in contempt that science which Plato reverenced. To such the *Manual* will make no appeal. But to those who feel a real interest in Greek thought, whatever be their angle of approach, this masterly account of what is perhaps the world's greatest intellectual adventure will be indispensable.

It only remains to say that during the ten years which have elapsed since the publication of the *History*, research work on the beginnings of mathematics has not stopped. Fresh light has been thrown on the problems of Egyptian mathematics by two magnificent editions of the Papyrus Rhind and by knowledge of the contents of the still unpublished Moscow Papyrus. This has enabled Sir Thomas Heath to improve upon the account of these problems given in the *History*. In other places, recent discoveries and studies have led to corrections, and in one instance to amplification, of earlier views.

It is difficult to avoid seeming fulsome in praise of this work. A mine of information, a delight to read, produced in a style worthy of its content, truly this is a book which every mathematician should buy and read.   T. A. A. B.

Math. Gazette, xi (1922-3), 348-351.

Advanced Calculus. By G. A. GIBSON. Pp. xvii + 510. 20s. net. 1931. (Macmillan.)

The readers of this book will experience a feeling of regret that Prof. Gibson did not live to see the completion of his work. At the time of his death in 1930, the book had been completely written, but had only partly passed through the press, and the rest of the task was undertaken by Prof. MacRobert.

Those who are at all familiar with Prof. Gibson's earlier works will expect this to be a rigorous, lucid and carefully written book, and they will not be disappointed. The book is really a comprehensive introduction to analysis, and may be described roughly as covering the same ground as the more elementary parts of Goursat's *Cours d'Analyse* and Bromwich's *Infinite Series*.

About half the book is devoted to differential calculus and the theory of infinite series. It begins with Dedekind's theory of irrational numbers and the more elementary theory of sets of points, followed by the differentiation of functions of one variable and of functions of several variables, considerable attention being paid to the problem of finding the derivatives after a change

of the variables. This leads naturally to a discussion of Jacobians and existence theorems concerning implicit functions. In the proofs of these existence theorems the conditions imposed are rather narrower than is necessary, but this is done deliberately in order to simplify the proofs as much as possible. Three chapters are devoted to infinite series and infinite products. A knowledge of the more elementary parts of the subject is assumed, and here are discussed such branches as absolute and conditional convergence, uniform convergence including Abel's theorem on the continuity of the sum of a power series, Tannery's theorems for series and products, and gamma functions.

The second half of the book begins with Riemann integration, and this is followed by discussions of curvilinear integrals and multiple integrals. An idea of the extent of the work will be conveyed by the statement that proofs are included of Green's theorem and Stokes' theorem, and the problem of changing the variables in a multiple integral is fully considered. There are two chapters on infinite integrals, which consider uniform convergence and the processes of differentiation and integration under the integral sign. The book closes with a chapter on the applications of the theory to the integration of series and to the gamma function, including Dirichlet's integrals and the asymptotic expansion of $\log \Gamma(x)$.

Apart from the actual text, the worked examples are chosen so as to convey a great deal of general information. Thus, in various parts of the book, we find such things as Rodrigues' formula and the recurrence formulae for the Legendre polynomials, the derivative of a determinant, Gauss's expression for the sum of the hypergeometric series with unit argument, and examples to illustrate Fourier's double integral theorem.

One point of historical interest must be mentioned. It is stated (on page 81) that Jacobi's transformation

$$\frac{d^{n-1} \cdot (1-x^2)^{n-\frac{1}{2}}}{dx^{n-1}} = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n} \sin n\theta,$$

where $x = \cos\theta$, was first given by Rodrigues in 1815. This is definitely in disagreement with a remark by Prof. G. N. Watson in his *Theory of Bessel Functions* (page 27) to the effect that the transformation has sometimes been erroneously attributed to Rodrigues, probably owing to an incorrect statement by Frenet that it was to be found in Rodrigues' works. Prof. Watson gives the history of the theorem with full references.

The book forms an excellent course of analysis suitable for the earlier part of an honours student's course at the university. A preliminary reading of some of the chapters would be a valuable training preparatory to the university course. W. N. B.

1. **A New Algebra for Schools.** By C. V. Durell. Parts I and II. Pp. x + 328. 4s. (or, without appendix, 3s.). 1930. (Bell.)

2. **A School Algebra.** By T. C. Batten and M. W. Brown. Part I. Pp. x + 198. 3s. 1930. (Murray.)

3. **Introductory School Algebra.** By G. W. Spriggs. Pp. vii + 120. 2s. 6d. 1930. (Pitman.)

4. **School Certificate Algebra.** By G. W. Spriggs. Pp. xviii + 326. 4s. 1930. (Pitman.)

1. Mr. Durell has drawn us still further into his debt by producing what is surely the ideal text-book. It has grown out of his previous writings, and in this respect affords a luminous reflection of current tendencies in the teaching of the elements of the subject.

For roughly the first two terms, represented by about 90 pages of the book, the subject is essentially "Generalized Arithmetic". The notation is explained as "shorthand", and justified by application to formulae. Equations become "Think-of-a-number" problems, and arithmetical ideas suffice to carry the pupil through elementary processes of multiplication, division, H.C.F., L.C.M., and manipulation of fractions, all involving monomials only.

Graphs of statistics, of loci and of travel problems are introduced, and an arithmetical treatment of brackets leads to the sign rules connected with their removal.

Only at this point do " positive and negative " numbers appear. They are treated from an essentially " vectorial " point of view and rules of signs thereby obtained. From now on, the subject develops along the usual lines. Complexity of example keeps pace with advance in ideas. Formula, equation, graph, and function are closely interwoven, and these notions made basic. It is interesting to note the germ of the calculus in a " constant gradient " treatment of the linear function. Geometrical illustrations illuminate factors and products and smooth the passage through " completing the square ". A two years' course closes with a discussion of literal equations (as transformation of formulae) and simultaneous quadratic equations.

If a text-book is to cater for that increase in manipulative skill which is so essential in elementary work, its examples must provide a touchstone of its value. In this respect, Mr. Durell leaves nothing to be desired. His wealth of examples, taken from geometry, physics and mechanics, are embarrassing in their interest, variety and ingenuity. They are mostly of a straightforward character, suited to the ordinary boy, and show a vivid appreciation of the ups and downs of the class-room. A valuable appendix (a unique feature of the work, inclusion of which is optional) affords ample scope for revision and for the brighter boy. Assuredly, Mr. Durell is an adept in the manufacture of the sugar-coated pill ; to change the metaphor, strange indeed must be the palate of the pupil for whom there is no cocktail here provided. We look forward to Part III which will complete the Certificate course.

2. The authors follow lines similar to those of Mr. Durell's work, " symbolic representation ", " formula " and " shorthand " being their watchwords from the outset. Directed numbers are presented in a novel and interesting manner. They are denoted by signs superposed, crutches which are later discarded in favour of the ordinary mode of progression. An expedient so transitory seems, however, of doubtful value as compared with the equally distinctive and more permanent " vector " brackets in normal use. Graphical work finds an adequate place, and is satisfactorily related with equations and their solution, and with functions, the functional notation being introduced. Factors are very happily dealt with from a geometrical point of view, and the work is carried up to the solution of the quadratic by all four methods.

The examples are numerous, pleasantly so in the earlier part of the book ; the problems are varied and consistently interesting. Historical notes interspersed add considerably to the value of the work, which (no small point) is beautifully printed. A later volume will completely cover the Certificate course.

3. Mr. Spriggs belongs to a different school. To him algebra is " essentially an analytical discipline, which should at every point present a challenge to the reason of the pupil ", a Spartan point of view which seems a precarious foundation for an elementary book, whose aim is to be read by the pupil. Directed numbers, with graphical illustration, appear at the outset, and the business of simple equation, brackets, fractions and factors, is somewhat formally transacted before any explicit correlation with arithmetic or generalisation by formula is attempted. The subject is carried up to the solution of the quadratic by factors. Summaries of " book work " form an interesting and useful feature. The examples are barely adequate ; more space might have been devoted to them, for surely the beginner prefers to " do " things rather than to read about them.

4. The author's aim has been to provide " a means of access to the spirit of the subject ", and he has certainly succeeded. . A revision chapter of " Fundamental Facts " formally sums up the apparatus. Then follow six chapters, each more or less self-contained, and all eminently readable. " The Formula " contains a discussion of manipulations and leads up to a sound treatment of variation of various kinds with their graphical representations. Then " The Equation " with discussion of simple, simultaneous (approached from a graphical point of view), quadratic, and simultaneous linear and

quadratic, varieties, with analytical methods of solution. "Functions" provides a fascinating and valuable chapter. The treatment follows naturally from that of the formula, and is essentially a graphical one. Linear, quadratic, cubic and reciprocal functions are fully dealt with, and properties of their graphs and methods of drawing them exhaustively discussed. Graphs provide an approach to the nature of the roots of a quadratic equation. The calculus is born in a capable treatment of gradient and carried on to a discussion of maxima and minima, the notions and conditions being derived from the plotting of the first and second derivatives along with the function itself. A treatment of implicit functions makes possible an introduction to the analytical geometry of the circle and conic sections. "The Number System" develops the ideas of surds, indices and logarithms, with graphical illustration. "Series" follow, with a pleasing geometrical presentation of convergence and "sum to infinity". A chapter on ratio and proportion, arrangements and groups, culminates in the Binomial Theorem with a positive integral index, and closes the work. Useful summaries are interspersed, and the problems are interesting. The book should stimulate the better pupils, though its method of presentation is hardly adapted to the "under-dogs" of the class-room.   E. L.

Teoría Geométrica de la Polaridad en las Figuras de Primera y Segunda Categoría. By J. Rey Pastor. Pp. viii + 294. Pesetas 20. 1929. (Rev. Mat. Hispano-Americana, Madrid.)

This is a systematic account, on purely geometrical lines, of the theory of polarity with respect to sets of points on a straight line and with respect to plane curves (there is no mention of three or more dimensions). It collects much scattered material : of this it contains a remarkably extensive bibliography. It is presumably not intended as a first introduction to the subject.

The preliminary chapter is an exposition of the work of Kötter, who received the Steiner prize of 1886 for his paper introducing the theory of plane curves without the use of analysis. It contains the definitions, and elementary properties, of involutions of sets of n points, and of plane curves of order n. In this chapter, as elsewhere, mathematical induction is used freely, both in definition and in demonstration ; e.g. a geometrical curve of order n is defined as the locus of the intersections of corresponding members in a pencil of curves of order $n-1$ and a related pencil of straight lines.

The second chapter deals with the fundamental properties of polarity : the first polar of a point P with respect to the n collinear points A is defined as the $n-1$ double points of the involution in which two sets are A, and P counted n times ; polarity with respect to a curve is deduced in the usual manner.

The outstanding features of the next portion of the work are the treatment of the Jacobian of three curves, the Hessian of a curve, and the Plücker formulae.

This is followed by a chapter on cubics and other special curves. Finally the significance of polarity in metrical geometry is considered.

The above describes only the skeleton of the work. Many interesting incidental results are included ; among them those by which the author brings the methods of other workers into relation with his own.

No attempt is made to base the argument on a minimum set of axioms. There is indeed some obscurity as to what are effectively the author's axioms ; and as to what limitation in this respect is imposed by the restriction to "geometrical" methods. Correspondence proofs are excluded as being equivalent to algebra. Some pains are taken in connection with imaginary elements, defined (following von Staudt) by means of real involutions. Slipshod proofs involving limiting positions are not allowed : Kötter is criticized for assuming that the limit of a certain system of algebraical curves is itself algebraical.

There are several noticeable misprints. An article by Kötter in the 40th Vol. of the *Jahrbuch über die Fortschritte der Mathematik* seems entirely irrelevant, though there are two distinct references to it : presumably the reference should be to the 25th Volume. There is unfortunately no alphabetical index. Mathematical Spanish is easy to read.   G. Timms.

Key to Advanced Trigonometry. By C. V. DURELL and A. ROBSON. Pp. 380. 15s. 1930. (Bell.)

Most teachers of advanced trigonometry will by this time have procured a copy of Messrs. Durell and Robson's book, which was favourably reviewed in the *Gazette* of last January. They will find this Key invaluable. The authors regard it as, to some extent, a supplementary teaching manual, and with this view the writer cordially agrees. Great pains have been taken to render the solutions clear, alternative methods are given in places, and there is hardly an exercise which does not receive a hint at least. But there is no prolixity. The book is printed in clear type and there appear to be few mistakes—a list of errata is given. The pages are headed with references to the corresponding exercises and pages of the main book, so that a teacher who is stumped by a hard example (and some are quite hard enough) can readily surmount the difficulty.

Plane Trigonometry. By F. G. W. BROWN. Pp. xii + 264. 4s. 1930. (Macmillan.)

The author's Progressive Trigonometry provides the usual numerical work and mensuration. This book is a sequel. It starts, however, from the beginning and contains a full treatment of trigonometry up to and including the properties of triangles and quadrilaterals. The last chapter, on " Limiting Values and Approximations," deals with such matters as the dip of the horizon and results like $\theta > \sin \theta > \theta - \frac{1}{6}\theta^3$. The scope is intended to coincide with that required for a School Certificate Examination. The author has certainly included all that is necessary and more. The treatment is in general good, the exercises are numerous and some quite sufficiently hard. The book is likely to prove of most use to young boys of special ability. There are a few blemishes which should be easily removable. On page 120, equations of the type $a \cos \theta + b \cos \phi = c$, $p \sin \theta + q \sin \phi = r$ are solved by eliminating ϕ; but the solutions are not verified and, for all the working shows, they might apply if the sign of q were changed. On page 62, the addition theorem is claimed to be completely established ; and if that means that the proof, as given, holds for angles of any size whatever, the reader may feel justifiably sceptical. Also on page 224, there are statements which amount to saying that if $f(n) < \phi(n)$ for all values of n and $f(n) \to l$, $\phi(n) \to L$ as $n \to \infty$, then $l < L$.

A Numerical Trigonometry. By B. C. MOLONY. Pp. 215. 3s. 1930. (Arnold.)

This book is attractive and well worth the money. Triangles of all kinds are solved before the addition formulae are introduced, several examples being worked out in the text. Much care is spent on mensuration and practical applications. There is a short section on the solution of equations by graphs, another on small angles, and the last chapter deals with compound angles, etc. In short the book gives quite a full treatment of all that is necessary for the non-specialist up to and including the solution of triangles by the usual formulae adapted for logarithmic work. There are numerous and varied exercises, five sets of revision papers, and answers are given.

Elementary Trigonometry. By A. F. VAN DER HEYDEN. Pp. vii + 164. 2s. 6d. 1930. (Rivingtons.)

A good little book, which should meet the needs of all but specialists. The scope coincides roughly with that of Mr. Molony's book, and the treatment is similar but not so full. No space is devoted to mensuration or small angles, while the graphs of the circular functions come in the last chapter. On the other hand, the author devotes a chapter to logarithms, which seems out of place. The book concludes with 70 test questions and answers to the exercises.

Test Papers in Trigonometry and Calculus. By J. J. WALTON. Pp. iv + 104. 2s. 6d. 1930. (Isaac Pitman.)

These papers, each of which contains six exercises, have been specially prepared for School Certificate and Matriculation purposes. Many will find them useful. There are 50 papers on trigonometry and 50 on calculus. The answers are not added.   C. J. A. T.